Newton's Law of Cooling:
The rate of change of the temperature of a body is proportional to the difference in temperature between the body and its surroundings.

Temperature of body \( T(t) \) dependent variable
time \( t \) independent variable
Temperature of surroundings \( T_0 \) constant

\[
\frac{dT}{dt} \propto T - T_0
\]

\[
\frac{dT}{dt} = -k(T - T_0), \quad k > 0
\]

Define a derivative of \( f(x) \) wrt \( x \)

\[
\frac{df}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}
\]

Right and left hand derivatives must be equal for \( f \) to be differentiable

\[
\lim_{h \to 0^-} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0^+} \frac{f(x+h) - f(x)}{h}
\]

E.g. \( f(x) = |x| \) not differentiable at \( x = 0 \)

\[
\frac{df}{dx} = f'(x) = \frac{d}{dx} f(x)
\]

\[
\frac{d}{dx} \left( \frac{df}{dx} \right) = \frac{d^2f}{dx^2} = f''(x) = f^{(2)}(x)
\]

Note: \( f'(2x) \) means \( \frac{df}{dx} \) with \( y = 2x \), \( \frac{dx}{dy} = \frac{1}{2} \frac{d}{dy} \left( \frac{d}{dx} \right) \)
\[ f(x) = o \left[ g(x) \right] \quad \text{as } x \to x_0 \]

If \( \lim_{x \to x_0} \frac{f}{g} = 0 \)

E.g. \( x = o \left( \frac{1}{x} \right) \quad \text{as } x \to 0 \)
\[ \frac{1}{x} = o \left( x \right) \quad \text{as } x \to \infty \]

\[ f(x) = O \left[ g(x) \right] \quad \text{as } x \to x_0 \]

"in of order"

If \( \frac{f(x)}{g(x)} \) is bounded as \( x \to x_0 \)

Then \( x = O(1) \quad \text{as } x \to 0 \)
\[ x = O \left( \frac{1}{x} \right) \quad \text{as } x \to 0 \]

Note \( f(x) = o \left[ g(x) \right] \Rightarrow f(x) = O \left[ g(x) \right] \) but not vice versa.

**Tangent line at \( x_0 \)**

\[
\frac{df}{dx} \bigg|_{x_0} = \frac{f(x_0 + h) - f(x_0)}{h} + \frac{o(h)}{h}
\]

\[ f(x_0 + h) = f(x_0) + h \frac{df}{dx} \bigg|_{x_0} + o(h) \]

Equation of tangent line at \( x_0 \) of \( y = f(x) \)

Replace \( x \) by \( x_0 + h \)

\[ y(x) = y_0 + m \left( x - x_0 \right) \]
\[ m = \frac{df}{dx} \bigg|_{x_0} = \frac{df}{dx} (x_0) \]
Recap:  
1) \( f(x) = \frac{d}{dx} \left[ g(x) \right] \) as \( x \to x_0 \)  
\( \Rightarrow \lim_{x \to x_0} \frac{f(x)}{g(x)} = 0 \)  

ii) \( f(x) = O \left[ g(x) \right] \) as \( x \to x_0 \) 
\( \Rightarrow \frac{f(x)}{g(x)} \) remains bounded 

iii) \( f(x_0 + h) = f(x_0) + h \frac{df}{dx} \bigg|_{x_0} + o(h) \)

**Chain rule**

Consider \( f(x) = F \left[ g(x) \right] \)  
\( \frac{df}{dx} = \lim_{h \to 0} \frac{F[g(x+h)] - F[g(x)]}{h} \)

\( = \lim_{h \to 0} \frac{1}{h} \left[ F[g(x)] + h \frac{dg}{dx} + o(h) \right] - F[g(x)] \)

\( = \lim_{h \to 0} \frac{1}{h} \left[ F[g(x)] + \left( h \frac{dg}{dx} + o(h) \right) F'[g(x)] + o(h) - F[g(x)] \right] \)

\( = \lim_{h \to 0} \left( \frac{dg}{dx} \times F'[g(x)] + o(h) \right) = \frac{d}{dx} \times F'[g(x)] \)

relied on finite \( \frac{dg}{dx} \) at the point in question and also \( \frac{dF}{dF} \) being finite. In other words, both the inner and outer functions must be differentiable.

\( \frac{d}{dx} \left[ \sin(x^2 - x + 2) \right] = \cos(x^2 - x + 2) \times (2x - 1) \)

**Product Rule**

\( f(x) = u(x) \cdot v(x) \), \( \frac{df}{dx} = u'v + uv' \)
Leibniz's rule

\[
\frac{d^n}{dx^n}(uv) = \sum_{k=0}^{n} \binom{n}{k} u^{n-k} v^k \frac{d^k}{dx^k}v
\]

where \( \binom{n}{k} = \frac{n!}{k!(n-k)!} \)

Taylor Series

Recall \( f(x+h) = f(x) + h f'(x) + o(h) \)

\[
f(x+h) = f(x) + h f'(x) + \frac{h^2}{2} f''(x) + \cdots + \frac{h^n}{n!} f^{(n)}(x) + E_n
\]

N.B. \( f \) must be \( n+1 \) times differentiable \( \Rightarrow \) (in complex plane, later)

then Taylor's Theorem states that

\( E_n = O(h^{n+1}) \) as \( h \to 0 \)

(20) \( E_n = o(h^n) \)

A Taylor series provides a local approximation to a function.

Contrast with a global approximation e.g. Fourier Series
Differential Equations 2

Alternative form

\[ f(x) = f(x_0) + (x - x_0) f'(x_0) + \ldots + \frac{(x - x_0)^n}{n!} f^{(n)}(x_0) + E_n \]

Taylor series of \( f(x) \) about the point \( x = x_0 \). A local approximation of the function near \( x_0 \).

Finding coefficients

WLOG, consider an expansion of \( f(x) \) about \( x = 0 \)

\[ f(x) = a_0 + a_1 x + a_2 x^2 + \ldots \]

\[ f'(x) = a_1 + 2a_2 x + \ldots \]

\[ f''(x) = 2a_2 + 3 \cdot 2a_3 x + \ldots \]

\[ f(0) = a_0 \]

\[ f'(0) = a_1 \]

\[ f''(0) = 2a_2 \]

\[ f'''(0) = 3 \cdot 2a_3 \]

\[ f^{(n)}(0) = n! \cdot a_n \]

\[ a_n = \frac{f^{(n)}(0)}{n!} \quad \text{QED} \]

L'Hopital's Rule

Suppose \( f(x) \) and \( g(x) \) are differentiable at \( x = x_0 \) and \( \lim_{x \to x_0} f(x) = \lim_{x \to x_0} g(x) = 0 \)

The limit \( \lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)} \), provided \( g'(x) \neq 0 \)

From Taylor Series (Linear part)

\[ f(x) = f(x_0) + (x - x_0) f'(x_0) + o(x - x_0) \]

\[ g(x) = g(x_0) + (x - x_0) g'(x_0) + o(x - x_0) \]

\[ \lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x_0)}{g'(x_0)} \]

3
Proof of L'Hopital's Rule

If $f$ and $g$ are continuous, differentiable at $x_0$, $g'(x) \neq 0$

\[
\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{g(x) - g(x_0)} = \lim_{x \to x_0} \frac{\frac{f(x) - f(x_0)}{x - x_0}}{\frac{g(x) - g(x_0)}{x - x_0}} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}
\]
Chain Rule: \( \frac{d}{dx} f(g(x)) = f'(g(x))g'(x) \)

Product Rule: \( \frac{d}{dx} (uv) = u v' + v u' \)

Taylor Series:
\[
 f(x) = f(x_0) + (x-x_0)f'(x_0) + \frac{(x-x_0)^2}{2!}f''(x_0) + \cdots + (x-x_0)^n \frac{f^{(n)}(x_0)}{n!} + O[(x-x_0)^{n+1}]
\]

L'Hopital's Rule:
\[
 \lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)} \quad \text{if } \lim f = \lim g = 0 \quad \text{and ratio limits exist}
\]

Integration:
An integral is a sum.
\[
 \int_a^b f(x) \, dx = \lim_{\Delta x \to 0} \sum_{n=0}^{N-1} f(x_n) \Delta x
\]

Area under the graph from \( x_n \) to \( x_{n+1} \):
\[
 \Delta A_n = f(x_n) \Delta x + O(\Delta x^2)
\]

Provided \( f \) is differentiable at \( x_n \)

Error in area:
\( O(\Delta x^2) \)

\( \bullet \) Area - error, \( O(x^3) \) using Taylor Series

Area from \( a \) to \( b \):
\[
 \lim_{N \to \infty} \left[ \sum_{n=0}^{N-1} f(x_n) \Delta x + O(N \Delta x^2) \right]
\]

Note: \( \Delta x = \frac{b-a}{N} \), \( N = \frac{b-a}{\Delta x} \) \( O(N \Delta x^2) = O(\Delta x) \)

\[
 \int_a^b f(x) \, dx
\]
Fundamental Theorem of Calculus

\[ \int_a^x f(t) \, dt \]

\[
\frac{dF}{dx} = \lim_{h \to 0} \frac{1}{h} \left[ \int_a^{x+h} f(t) \, dt - \int_a^x f(t) \, dt \right]
\]

\[
= \lim_{h \to 0} \frac{1}{h} \left[ f(x)h + O(h^2) \right]
\]

Wrong notation:

Notation \( F(x) = \int_a^x f(x) \, dx \int_0^x f(t) \, dt \)

Similarly:

\[
\frac{d}{dx} \int_a^x f(t) \, dt = -f(x)
\]

\[
\frac{d}{dx} \int_a^x f(t) \, dt = f(g(x)) g'(x)
\]

Integration by substitution

Integration is an act of recognition. If the integrand contains a function of a function, it can sometimes aid recognition to substitute for the outer function. Especially helpful if we can recognize the structure of the chain rule.

\[
\int 1 - 2x \sqrt{5x - x^2} \, dx = x - x^2
\]

\[
\frac{du}{\sqrt{5x - x^2}} \quad du = (1 - 2x) \, dx
\]

\[
= \int \frac{du}{\sqrt{5x - x^2}} = 2\sqrt{5x - x^2} + C
\]

Trigonometric substitution
Differential Equations

Trigonometric Substitutions

Useful Identities

<table>
<thead>
<tr>
<th>Substitution</th>
<th>( x = \sin \theta )</th>
<th>( dx = \cos \theta , d\theta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \cos^2 \theta + \sin^2 \theta = 1 )</td>
<td>( \frac{1}{1-x^2} )</td>
<td>( 1+x^2 )</td>
</tr>
<tr>
<td>( 1 + \tan^2 \theta = \sec^2 \theta )</td>
<td>( \sqrt{1+x^2} )</td>
<td>( \frac{\sqrt{x^2-1}}{x} )</td>
</tr>
<tr>
<td>( \cosh^2 u - \sinh^2 u = 1 )</td>
<td>( \sqrt{1+x^2} )</td>
<td>( \frac{\sqrt{x^2-1}}{x} )</td>
</tr>
<tr>
<td>( 1 - \tanh^2 u = \text{sech}^2 u )</td>
<td>( \sqrt{1+x^2} )</td>
<td>( \frac{\sqrt{x^2-1}}{x} )</td>
</tr>
</tbody>
</table>

\[
\int \frac{2x}{x^2-1} \, dx = \int \frac{2x}{1+x^2} \, dx = \int \frac{1}{1-x(1-x)} \, dx
\]

\[
x-1 = \sin \theta, \quad x = 1 + \sin \theta
\]

\[
dx = \cos \theta \, d\theta
\]

\[
x-1 = \sin \theta, \quad x = 1 + \sin \theta
\]

\[
dx = \cos \theta \, d\theta
\]

\[
= \frac{1}{2} (x-1)^2 + \frac{1}{2} \arcsin(x-1) + C
\]

By Parts

<table>
<thead>
<tr>
<th>Product rule</th>
<th>((uv)' = u'v + uv')</th>
</tr>
</thead>
</table>

\[
\int uv' \, dx = \int (uv)' - u'v \, dx = uv - \int u'v \, dx
\]

\[
\text{e.g.}, \quad \int xe^{-x} \, dx\]

\[
u(x) = x, \quad u'(x) = e^{-x}
\]

\[
\int xe^{-x} \, dx = \left[ xe^{-x} \right]_0^\infty - \int e^{-x} \, dx
\]

\[
= e^{-x} \bigg|_0^\infty - \int e^{-x} \, dx = 1
\]

\[
\int \ln x \, dx\]

\[
u(x) = \ln x, \quad u'(x) = \frac{1}{x}
\]

\[
v = x
\]

\[
\text{or inverse sin x + C}
\]

\[
\frac{1}{x} \ln x - \int \frac{1}{x} \, dx = x \ln x - x + C
\]
Functions of several variables

Partial differentiation:
Consider a function \( f(x, y) \)
- e.g. height of terrain / hill
  - pressure (temperature)
  - density of a gas

\[\begin{align*}
\text{function of east/west/north/south coordinates} & = f(\text{temp, pressure}) \\
\end{align*}\]

Represent such functions either on a graph:

\[\begin{align*}
\text{or as a contour plot} \\
\text{contours, curves along which} \quad f = \text{constant}
\end{align*}\]

Q: What is the slope of a hill?
A: Depends which direction you are facing.

Begin by finding the slope in directions parallel to the axes.
The partial derivative of \( f(x, y) \) wrt \( x \)
\[\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}\]

Similarly \( \frac{\partial f}{\partial y} \) wrt \( y \)
\[\frac{\partial f}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}\]

Calculating partial derivatives:
\( f(x, y) = x^2y + y^3 + e \)
\[\begin{align*}
\frac{\partial f}{\partial x} & = 2xy + y^2e \\
\frac{\partial f}{\partial y} & = 3x^2 + 2xye \\
\end{align*}\]

Can also find 2nd partial derivatives
\[
\frac{d^2 f}{dx^2} = 2 + y^4 e^{xy^2} \\
\frac{d^2 f}{dy^2} = 6y + 2xe^{xy^2} + 4x^2 y^2 e^{xy^2}
\]

\[
\frac{d^2 f}{dx dy} = \frac{d}{dx} \left( \frac{df}{dy} \right) \\
= 2ye^{xy^2} + 2xy^3 e^{xy^2}
\]

\[
\frac{d^2 f}{dy dx} = 2ye^{xy^2} + 2xy^3 e^{xy^2}
\]

It is a general rule (in Euclidean space) that \( \frac{d^2 f}{dx dy} = \frac{d^2 f}{dy dx} \).

\text{NB}

To be careful, we indicate which variable or variables are being held constant, but if no indication we assume everything is constant except the variable we are differentiating with respect to.

e.g. \( f = f(x, y, z) \)

\[\frac{df}{dx} = \frac{df}{dx}_{y, z} \neq \frac{df}{dy} \text{ in which } z \text{ may vary} \]

Alternative notation \( f_x = \frac{df}{dx} \quad f_{xy} = \frac{d^2 f}{dy dx} \)

\text{Chain rule}

\[
f' = f(x + 5x, y + 5y) - f(x, y) \\
= f(x + 5x, y + 5y) - f(x + 5x, y) + f(x + 5x, y) \\
\]

\[
f' = 5 \frac{df}{dy}(y + 5y) + o(5y) \\
+ 5 \frac{df}{dy}(y) + o(5x)
\]
Differential Equations

\[ \frac{df}{dy} (x, y) + o(5x) \int sy + o(5y) \]
\[ + \frac{df}{dx} (x, y) s\hat{x} + o(5\hat{x}) \]

(*) \( df = \frac{df}{dx} (x, y) s\hat{x} + \frac{df}{dy} (x, y) s\hat{y} + o(5\hat{x}, 5\hat{y}) \)

Take limit as \( s\hat{x} \to 0, s\hat{y} \to 0 \)

\( df = \frac{df}{dx} dx + \frac{df}{dy} dy \)

This is the chain rule in differential form. We understand it as a shorthand for (*) knowing that we shall either sum terms or divide by another infinitesimal quantity before taking the limit.

E.g. \( \int \int df = \int \int \frac{df}{dx} dx + \int \int \frac{df}{dy} dy \)

Along a path, \( (x, y) = [x(t), y(t)] \)

where \( t \) is a parameter along the path (e.g. time)

\( f(x, y) = f[x(t), y(t)] \)

\[ \frac{df}{dt} = \lim_{\rho \to 0} \frac{df}{dt} = \lim_{\rho \to 0} \left[ \frac{df}{dx} \frac{dx}{dt} + \frac{df}{dy} \frac{dy}{dt} + o\left(\frac{5x, 5y}{\rho}\right) \right] \]

\[ \frac{df}{dt} = \frac{df}{dx} \frac{dx}{dt} + \frac{df}{dy} \frac{dy}{dt} \]

Chain rule
Differential Equations 5

Recap
\[
\frac{df}{dy} (x + ax, y) \Delta y = \left[ \frac{df}{dy} (x, y) + \frac{df}{dx} \Delta x + o(\Delta x) \right] \Delta y
\]

\[x(t), y(t)\]

Chain Rule

Short for
\[
\frac{df}{dx} = \frac{df}{dx} \Delta x + \frac{df}{dy} \Delta y
\]

Along a path:
\[
\frac{df}{dt} = \frac{df}{dx} \frac{dx}{dt} + \frac{df}{dy} \frac{dy}{dt}
\]

Path can also be given
\[D = y(x)\]

\[f = f \left[ x(t), y(x) \right]\]

Change of variables:
\[e.g. x = (r, \theta), \ y = (r, \theta)\]

\[f = f \left[ x(1, \theta), y(1, \theta) \right]\]

\[\frac{df}{dr} |_{r=1} = \frac{df}{dx} \frac{dx}{dr} |_{r=1} + \frac{df}{dy} \frac{dy}{dr} |_{r=1}\]

Similarly
\[\frac{df}{d\theta} |_{\theta=0} = \frac{df}{dx} \frac{dx}{d\theta} |_{\theta=0} + \frac{df}{dy} \frac{dy}{d\theta} |_{\theta=0}\]

Implicit Differentiation:
\[F(x, y, z) = \text{constant}\]

It implicitly defines
\[z = z(x, y)\]

or\[x = x(y, z)\]

or\[y = y(x, z)\]

\[e.g. xy^2 + yz^2 + z^5x = 5\]

Solve for \(x\):
\[x = \frac{5 - yz^2}{y^2 + zs}\]

... explicitly

Could also find \(y = (x, z)\) by solving the quadratic \(\Rightarrow\) function with two branches but we cannot find \(z = z(x, y)\), would have to solve a quartic

Find \(\frac{dz}{dx}\) by differentiating w.r.t. \(x\), holding \(y\) constant:
\[y^2 + 2yz + 5z^4x \cdot \frac{dz}{dx} |_{y=0} + z^5 = 0\]

\[\frac{dz}{dx} |_{y=0} = -\frac{z^5}{y^2 + z^5x}\]
In general, think of \( F(x, y, z(x, y)) = \) constant

Chain rule in differential form:
\[
dF = \frac{\partial F}{\partial x} \, dx + \frac{\partial F}{\partial y} \, dy + \frac{\partial F}{\partial z} \, dz
\]

\[
\frac{\partial F}{\partial x} \bigg|_y = \frac{\partial F}{\partial x} \frac{\partial x}{\partial x} \bigg|_y + \frac{\partial F}{\partial y} \frac{\partial y}{\partial x} \bigg|_y + \frac{\partial F}{\partial z} \frac{\partial z}{\partial x} \bigg|_y = 0
\]

\[
\frac{\partial F}{\partial x} \bigg|_y = \frac{\partial F}{\partial z} \frac{\partial z}{\partial x} = 0
\]

\[
\frac{\partial z}{\partial x} \bigg|_y = -\frac{\frac{\partial F}{\partial y} \frac{\partial z}{\partial x} \bigg|_y}{\frac{\partial F}{\partial x} \bigg|_y} = -\frac{\frac{\partial z}{\partial x} \bigg|_y}{\frac{\partial F}{\partial x} \bigg|_y}
\]

Similarly:
\[
\frac{\partial z}{\partial y} \bigg|_x = -\frac{\frac{\partial z}{\partial x} \bigg|_y}{\frac{\partial z}{\partial y} \bigg|_x}
\]

Note: Normal rules apply provided the same variables are being held constant.
\[(x, y) \rightarrow (r, \theta)\]

\[
\frac{\partial r}{\partial x} = \frac{\partial x}{\partial r} \quad \text{because} \quad \frac{\partial r}{\partial x} \bigg|_y \neq \frac{\partial x}{\partial r} \bigg|_y
\]

But \( \frac{\partial r}{\partial x} \bigg|_y = \frac{\partial x}{\partial r} \bigg|_y \quad \checkmark \)

Differentiation of an integral: with respect to a parameter
Consider a family of functions \( f(x, c) \)

Define a function \( I(b, c) = \int_{0}^{b} f(x, c) \, dx \)

\[
\frac{\partial I}{\partial c} \bigg|_{c} = f(b, c) \quad \text{by fundamental theorem of calculus}
\]

\[
\frac{\partial I}{\partial b} \bigg|_{b} = \lim_{\Delta c \to 0} \frac{1}{\Delta c} \left[ \int_{0}^{b} f(x, c+\Delta c) \, dx - \int_{0}^{b} f(x, c) \, dx \right]
\]
\[
\lim_{n \to 0} \int_{a}^{b} \frac{f(x, c+sc) - f(x, c)}{sc} \, dx
\]

Consider:
\[
I[b(x), c(x)] = \int_{a}^{b} f[y, c(x)] \, dy
\]
\[
I(b) = \int_{0}^{b} e^{-ax^{2}y} \, dy
\]
\[
\frac{dI}{dx} = f(b, c) \frac{db}{dx} + \frac{dc}{dx} \int_{0}^{b} \frac{f(y, c)}{ac} \, dy
\]

\[
\frac{dI}{dx} = e^{-x^{3}} + \int_{0}^{x} -2xy e^{-x^{2}y} \, dy
\]
Differential Equations

\[ \int b(x), c(x) \right] = \int f[y, c(x)] dy \]

\[
\frac{dT}{dx} = \frac{dT}{db} \frac{db}{dx} + \frac{dT}{dc} \frac{dc}{dx}
\]

\[
= f(b, c) \frac{db}{dx} + \frac{dc}{dx} \int_0^b \frac{df}{dc} \frac{dy}{y} dy
\]

\[
i) I = \int e^{-x^2} dx, \quad \frac{dI}{dx} = e^{-x^2}
\]

\[
ii) I = \int e^{-x^2} dx, \quad \frac{dI}{dx} = \int_0^1 -x^2 e^{-x^2} dx
\]

\[
iii) I = \int e^{-x^2} dx
\]

Exponential Function \( f(x) = a^x \), \( a > 0 \), \( a \) is constant

\[
\frac{df}{dx} = \lim_{h \to 0} \frac{a^{x+h} - a^x}{h} = \lim_{h \to 0} a^x \frac{a^h - 1}{h} = a^x \lim_{h \to 0} \frac{a^h - 1}{h} = a^x \cdot \text{constant} = f'(0)
\]

\[
\frac{df}{dx} = f(x) \text{ where } f = \lim_{h \to 0} \frac{a^h - 1}{h} = \text{constant} = f'(0)
\]
Define \( f(x) = \exp(x) = e^x \) by \( \frac{df}{dx} = f(x) \) with \( f(0) = 1 \).

Proof that \( e = \lim_{k \to \infty} (1 + \frac{1}{k})^k \) is on example sheet.

\[ y = a^x = e^{x \ln a}, \text{ then } \frac{dy}{dx} = \ln a \cdot e^{x \ln a} = \ln a \cdot e^x \]

\[ \lambda = \ln a \]

First order, linear differential equations.

\[ \frac{d}{dx} (e^{\lambda x}) = \lambda (e^{\lambda x}) \]

\( e^x \) is an eigenfunction of the differential operator \( \frac{d}{dx} \).

The functional form is unchanged by the operator, only the magnitude is changed.

Any linear homogeneous ordinary differential equation with constant coefficients has solutions of the form \( e^{\lambda x} \).

E.g. \( 5y' - 3y = 0 \)

Linear, the dependent variable appears only linearly.

\[ x y' + y = e^x \]

is linear,

\[ y y' + xy = 5 \]

is non-linear.

Homogeneous \( y = 0 \) is a solution.

Constant Coefficients: independent variable does not appear explicitly.

First order. No higher derivatives than 1st are involved.

\[ y = e^{\lambda x} \]

In example \( 5y' - 3y = 0 \), \( \lambda = \frac{3}{5} \)

As \( e^{\lambda x} \neq 0 \) so \( y = e^{3x} \) is a solution.
Differential Equations

1) Because the equation is linear and homogeneous, any multiple of a solution is also a solution.
   \[ y = Ae^{\frac{3}{5}x} \] is also a solution for any constant \( A \).

2) An \( n \)th order linear differential equation has (only) \( n \) independent solutions. Therefore \( y = Ae^{\frac{3}{5}x} \) is the most general solution to (1).

Can determine \( A \) by applying a boundary condition, i.e., \( y(x) \) at \( x = a \).

Discrete Equation

\[ S_y' - 3y = 0, \quad y = y_0 \quad \text{at} \quad x = a \]

Approximate by \( \frac{S_{yn+1} - y_n}{h} = 3y_n \) with \( y(0) = y_0 \) and \( x_n = nh \) (compound interest formula).

\[ y_{n+1} = (1 + \frac{3h}{5})y_n \]

Repeatedly \( y_n = (1 + \frac{3h}{5})^n y_0 \)

Take limit as \( n \to \infty \):

\[ y(x) = \lim_{n \to \infty} y_0 \left( 1 + \frac{3x}{5n} \right)^n \]

\[ = y_0 e^{\frac{3x}{5}} \]
**Differential Equations**

Series solution

Try a solution of the form: \( y = \sum_{n=0}^{\infty} a_n x^n \)

\[ y' = \sum_{n=0}^{\infty} a_n x^{n-1} \quad 5y' - 3y = 0 \]

\( 5(xy') - 3xy = 0 \Rightarrow \sum a_n [5n - 3x]x^n = 0 \)

Coefficient of \( x^n \):

\( 5n a_n - 3a_{n-1} = 0 \)

\( n = 0 \quad 0 \cdot a_0 = 0 \Rightarrow a_0 \) is arbitrary

\( n > 0 \)

\[ a_n = \frac{3n}{5n} a_{n-1} = \frac{(3/5)^n}{n!} a_0 \]

\[ y = a_0 \sum_{n=0}^{\infty} \frac{(\frac{3}{5})^n}{n!} x^n = a_0 e^{\frac{3}{5}x} \]

**Forced equations - Inhomogeneous**

1) Constant forcing \( 5y' - 3y = 10 \)

Can spot a steady (equilibrium) solution: \( y = y_0 = -\frac{10}{3} \quad \Rightarrow \quad y_0' = 0 \)

Write \( y = y_p + y_c \)

\[ y = -\frac{10}{3} + Ae^t \]

2) Eigenfunction forcing. In a radioactive rock, isotope \( A \) decays into isotope \( B \) at a rate proportional to the number, \( a \), of remaining nuclei of \( A \), and \( B \) decays to \( C \), at a rate proportional to the number \( b \), of remaining nuclei of \( B \).

\[ \frac{da}{dt} = -ka a \quad \frac{db}{dt} = -kb b \]

\[ \frac{da}{dt} + ka a = 0 \quad \frac{db}{dt} = -k_a a_0 e^{-kt} - k_b b \]

\[ a = a_0 e^{-kt} \quad \frac{db}{dt} + k_b b = -k_a a_0 e^{-kt} \]

Note: forcing is an eigenfunction of the differential operator on the LHS by a particular integral:

\[ b_0 = C e^{-kt} \Rightarrow -ka C + k_b C = k_a a_0 \Rightarrow C = \frac{k_a a_0}{k_b - k_a} \]

provided \( ka \neq k_b \)
Write \( b = b_0 + b_1 \)
\[
\frac{b_1}{k_b b_c} = 0 \\
\frac{b_c}{D_e} = 0
\]
\[
b = \frac{k_a}{k_b - k_a} a_0 e^{-k_a t} + D e^{-k_b t}
\]

Suppose \( b = 0 \) at \( t = 0 \)
\[
b = \frac{k_a}{k_b - k_a} a_0 \left( e^{-k_a t} - e^{-k_b t} \right)
\]
\[
\frac{b}{a} = \frac{k_a}{k_b - k_a} \left[ 1 - e^{(k_b - k_a)t} \right]
\]
This allows a rock to be dated from the relative proportions of certain isotopes.

Non-constant coefficients

General form: \( a(x) y' + b(x) y = c(x) \)

Divide by \( a(x) \) to get standard form: \( y' + p(x) y = f(x) \)

Multiply by \( A(x) \) so \( A(x) y' + \mu_0 y = \mu_0 f \)

\[
\mu_0 = \frac{\mu'}{\mu}, \quad \int \mu_0 \, dx = \int \frac{\mu'}{\mu} \, dx = \ln \mu
\]

\( A(my)' = A \mu f, \quad A - my = \int \mu f \, dx \) etc.

\( \text{e.g.} \quad xy' + (1-x)y = 1, \quad y' + (\frac{x}{x-1})y = \frac{1}{x} - x \)

If \( \mu = \exp \left( \int \frac{1}{x} - 1 \, dx \right) = e^{\ln x - x} = xe^{-x} \)

\[
(xe^{-x} y)' = e^{-x} \]

\[
x e^{-x} y = -e^{-x} + c \quad y = \frac{c - e^{-x}}{xe^{-x}} = -\frac{1}{x} + \frac{x}{e^x}
\]
Nonlinear First Order

In general, a first order ordinary differential equation has the form

$$ Q(x, y) \frac{dy}{dx} + P(x, y) = 0 $$

Separable Equations

The equation is separable if it can be manipulated into the form

$$ q(y) \, dy = p(x) \, dx $$

in which case, the solution can be found by integration

$$ \int q(y) \, dy = \int p(x) \, dx $$

E.g.,

$$ (6c^2y - 3y) \, \frac{dy}{dx} - 2x \, y^2 = 4x $$

$$ \frac{dy}{dx} = \frac{4x + 2xy^2}{x^2y - 3y} = \frac{2x(2+y^2)}{y(x^2-3)} $$

$$ \int \frac{y}{2+y^2} \, dy = \int \frac{2x}{x^2-3} \, dx $$

$$ \frac{1}{2} \ln(2+y^2) = \ln(x^2-3) + C $$

$$ (2+y^2)^{\frac{1}{2}} = A(x^2-3) $$

Exact Equations

$$ Q(x, y) \frac{dy}{dx} + P(x, y) = 0 $$

is an exact equation if and only if

$$ Q(x, y) \, dy + P(x, y) \, dx $$

is an exact differential of a function $f(x, y)$, i.e.,

$$ \exists f(x, y), df \neq P \, dx + Q \, dy $$

in which case, $df = 0$ from the differential equation, so $f = const.$

Suppose there exists such a function $f(x, y)$

Chain Rule

$$ df = \frac{df}{dx} \, dx + \frac{df}{dy} \, dy $$

$$ \frac{df}{dx} = P, \quad \frac{df}{dy} = Q $$

Note

$$ \frac{df}{dy} \frac{dy}{dx} = \frac{df}{dx} \frac{dy}{dx} = \frac{df}{dx} = P $$

$$ \frac{df}{dx} \frac{dy}{dx} = \frac{df}{dx} \frac{dy}{dx} = \frac{df}{dx} = P $$

$$ \frac{df}{dy} = \frac{df}{dx} $$
True (proof not given) that if \( \frac{\Delta P}{\Delta x} = \frac{\Delta Q}{\Delta y} \) throughout a simply connected domain \( D \) the Pdx + Qdy is an exact differential of a single valued function \( f(x, y) \) in \( D \).

**Note**

1. The reverse implication follows locally from the chain rule.
2. What is a simply connected domain? "A domain with no holes"

(a) Slice of Swiss cheese  (b) A whole Swiss cheese

Example

\[
b (y - x) \frac{dy}{dx} + (2x - 3y^2) = 0
\]

\[
(2x - 3y^2) dx + b(y - x) dy = 0
\]

\[
p = 2x - 3y^2 \quad Q = b(y - x)
\]

\[
\frac{\partial P}{\partial y} = -6y \quad \frac{\partial Q}{\partial x} = -6y
\]

\[
\frac{\Delta f}{\Delta x} = 2x - 3y^2 \quad \frac{\Delta f}{\Delta y} = b(y - x) = -6xy + 6y
\]

\[
f = x^2 - 3xy^2 + g(y) \quad \Rightarrow \frac{\Delta f}{\Delta y} = -6xy + g'(y)
\]

\[
g = 2y^3 + C \quad f = x^2 - 3xy^2 + 2y^3 + C
\]
Solution of the equation is $f = \text{constant}$

$x^2 - 3xy^2 + 2y^3 + C = \text{constant}$

E.g. \( \frac{dy}{dt} = t(1 - y^2) \)

\[
\int \frac{1}{1 - y^2} \, dy = \int t \, dt
\]

\[
\tanh^{-1} y = \frac{1}{2} t^2 + C
\]

\[
y = \tanh \left( \frac{1}{2} t^2 + C \right)
\]

If we have an initial condition, we can determine $A$.

E.g. if $y(0) = 0$, $A = 1$. 

\[
y = \frac{A - e^{-t^2}}{A + e^{-t^2}}
\]
Differential Equations

\[ y' = t(1 - y^2) \], generally \[ \frac{dy}{dt} = F(y, t) \]

Note first that \( y' = 0 \) where \( y = \pm 1 \) or \( t = 0 \). Note also that \( \frac{dy}{dt} > 0 \) for \(-1 < y < 1, t > 0\)
and negative for \( y < -1, y > 1, t > 0\)

Consider the contours of \( f \) which are called isoclines, of the differential equati
\[ t \times (1 - y^2) = c \]
\[ t = \frac{1 - y^2}{(1 + y)(1 - y)} \]

Note that as \( |y| \to \infty \), \( \frac{dy}{dt} \to -y^2 \)
\[ \frac{dy}{y^2} = t \, dt \]
\[ \frac{1}{y^2} \, t = \frac{1}{2} t^2 - 0 \]

Note if \( f(y, t) \) is single valued curves do not cross. \( y = \frac{t^2 - 0}{2} \), \( y = 1 \) is a stable attractor, \( y = -1 \) is an unstable attractor.

Equilibrium and stability
Fixed points (equilibrium points) are where \( \frac{dy}{dt} = 0 \) for all \( t \)
\[ f(y, t) = 0 \] for all \( t \). In our example there are \( y = \pm 1 \).
We can see from the solution curves that as time increases, solutions converge towards \( y = +1 \), a stable fixed point but diverge from \( y = -1 \), an unstable fixed point.
Perturbation analysis - to determine stability and nature of solutions.

\[ \frac{dy}{dt} = f(y, t), \quad y \text{ is a fixed point, i.e. } f(y, t) = 0 \]

Write \( y = a + \epsilon(t) \), \( \epsilon \) perturbation.

Substitute: \( \frac{d\epsilon}{dt} = f(a + \epsilon, t) \)

\[ \frac{d\epsilon}{dt} = f(a, t) + \epsilon \frac{\partial f}{\partial y} (a, t) + O(\epsilon^2) \]

\[ \Rightarrow \frac{d\epsilon}{dt} = [\frac{\partial f}{\partial y}]_a \epsilon \quad \text{linear equation} \]

In example \( f = t(1-y^2) \)

\[ \frac{\partial f}{\partial y} = -2yt \]

Near \( y = 1 \), \( \epsilon = -2t \epsilon \)

\[ \epsilon = E_0 e^{-t^2} \quad \Rightarrow \quad y = 1 \text{ is stable.} \]

This is true for sufficiently small \( E_0 \).

Near \( y = -1 \), \( \epsilon = 2t \epsilon \)

\[ \epsilon = E_0 e^{t^2} \quad \Rightarrow \quad y = -1 \text{ grows to infinity} \quad \Rightarrow \quad y = -1 \text{ is unstable.} \]

Autonomous Systems

\[ \dot{y} = f(y), \text{ independent of } t. \] Then, near a fixed point \( y = a \), \( f(a) = 0 \)

write \( y = a + \epsilon(t) \Rightarrow \epsilon = \frac{df}{dy}(a) \cdot \epsilon \quad \Rightarrow \epsilon = E_0 e^{k t} \)

Fixed point is stable or unstable according to whether \( \frac{df}{dy}(a) \) is +ve or -ve.

(stable if +ve, unstable if +ve)
Example - Chemical reaction kinetics.

\[ \text{NaOH} + \text{HCl} \rightarrow \text{H}_2\text{O} + \text{NaCl} \]

\[ \text{water} \]

\[ \text{molea}_{a} \; \text{b} \; \text{c} \; \text{c} \]

Initially: \( a = a_0 \), \( b = b_0 \), \( c = 0 \)

If the reactants are in dilute solution (e.g. water) then the reaction rate is linear in both \( a \) and \( b \).

\[ \frac{dc}{dt} = \lambda (ab) \quad \text{for some } \lambda \]

\[ \frac{dc}{dt} = \lambda (a_0 - c)(b_0 - c) \]

\[ \dot{y} = f(y) \]

\[ y(t) = a + \varepsilon(t) \quad f(a) = 0, \text{ a fixed point} \]

\[ \varepsilon = \frac{d\varepsilon}{dt} = \frac{dy}{dt} = f(a + \varepsilon) \rightarrow f(a) + \varepsilon \frac{df}{dy} a \]

\[ \varepsilon = \frac{df}{dy} a \]

\[ \Rightarrow \text{Stability for } \frac{df}{dy} a \rightarrow -\text{ve} \]

\[ \text{Instability for } \frac{df}{dy} a \rightarrow +\text{ve} \]
Differential Equations

\[ \text{Na}_2\text{OH} + \text{HCl} \rightarrow \text{H}_2\text{O} + \text{NaCl} \]

Initially

\[ a_0 \quad b_0 \quad 0 \quad 0 \]

\[ \frac{dc}{dt} = \lambda ab = \lambda (a_0 - c)(b_0 - c) = f(c) \]

We can plot \( \frac{dc}{dt} \) as a function of \( c \)

\[ \frac{dc}{dt} = f \]

Determine the phase portrait. The dimension of the relevant phase space is equal to the order of the differential system.

Phase portrait

Arrows point in the direction of increasing \( t \). From the phase portrait we can see easily that \( c = a_0 \) is a stable fixed point, \( c = b_0 \) an unstable fixed point.

Exercise, show \( c = \frac{a_0 b_0}{b_0 - a_0 e^{-\lambda(b_0-a_0)t}} \)

Logistic equation - A simple model of population dynamics

Population \( y \), birth rate \( a \), death rate \( \beta \): \( \frac{dy}{dt} = (a - \beta) \cdot y \Rightarrow y = y_0 \cdot e^{(a-\beta)t} \)

Population increases or decreases exponentially depending whether birth rates exceed death rates.
Fighting for limited resources

Probability of some food being found $X \cdot Y$

same food being found by two individuals $X \cdot Y^2$

If food is scarce, then "fight to the death".

Death rate due to fighting $X \cdot Y^2$, $r = \alpha - \beta$

\[
\frac{du}{dt} = (\alpha - \beta)Y - \gamma Y^2, \quad u = \gamma Y \left(1 - \frac{y}{Y}\right) \quad Y = \frac{r}{r}
\]

differential logistic equation

Phase portrait

Intermediate

\[ f(x) \]

\[ \begin{array}{c}
\downarrow \\
\rightarrow
\end{array} \]

\[ y \]

\[ \leftarrow \\
\uparrow \text{unstable} \quad \text{stable} \]

When population is small, $u \approx \gamma y$, no competition, exponential growth.
Eventually, a stable equilibrium $y = \dot{y}$ is reached.

Discrete Equations: Evolution of species may occur discretely (e.g., birth in spring, death in winter) rather than continuously. So a better model might be

\[ x_{n+1} = \lambda x_n (1 - x_n) \]

Discrete logistic equation, or difference map. $x_{n+1} = f(x_n)$

Behaviour

\[ \lambda < 1 \]

\[ x_{n+1} = x_n \]

\[ x_n = 0 \quad x_n = 1 \quad \frac{1}{\lambda} \]

From picture, $x = 0$ is a stable fixed point.
Stability: suppose \( x_n = x \) is a fixed point

Write \( x_n = x + E_n \) perturbation

\[
X + E_{n+1} = f(X + E_n)
\]

\[
X + E_{n+1} = f(x) + E_n f'(x) + O(E_n^2)
\]

\[
E_{n+1} = \frac{f'(x)}{2} E_n
\]

Fixed point is stable if \( |\frac{E_{n+1}}{E_n}| < 1 \) for all \( n \)

\[
|f'(x)| < 1
\]

For logistic equation

\[
f = \lambda x(1-x)
\]

\[
f' = \lambda - 2\lambda x
\]

\( x = 0, f' = \lambda \), so \( x = 0 \) is stable \( \Rightarrow |\lambda| < 1 \)

\( x = 1 - \frac{1}{\lambda} \) is stable if \( |\lambda - 2\lambda + 2| < 1 \) \( \Rightarrow |2 - \lambda| < 1 \)

\[
\frac{E_{n+1}}{E_n} = f'(x) = 2 - \lambda, > 0 \text{ for } \lambda < 2
\]

\( < 0 \text{ for } \lambda > 2 \)
Relationship between logistic equation and logistic map

Logistic equation:

\[ \frac{dy}{dt} = ry \left(1 - \frac{y}{Y}\right). \]

Approximate the left-hand side to give

\[ \frac{y_{n+1} - y_n}{\Delta t} \approx ry_n \left(1 - \frac{y_n}{Y}\right) \]

\[ \Rightarrow y_{n+1} \approx y_n + r\Delta t y_n \left(1 - \frac{y_n}{Y}\right) \]

\[ = (1 + r\Delta t)y_n - \frac{r\Delta t}{Y} y_n^2 \]

\[ = (1 + r\Delta t)y_n \left[1 - \left(\frac{r\Delta t}{1 + r\Delta t}\right) \frac{y_n}{Y}\right] \]

Write

\[ \lambda = 1 + r\Delta t, \quad x_n = \left(\frac{r\Delta t}{1 + r\Delta t}\right) \frac{y_n}{Y} \]

Then

\[ x_{n+1} = \lambda x_n (1 - x_n), \]

which is the logistic map.
\[ x_{n+1} = \lambda x_n(1-x_n) \]

\( \lambda < 1 \)

\( 1 < \lambda < 2 \)

\( 2 < \lambda < 3 \)

\[ 3 c \lambda < 1 + \sqrt{6} \approx 3.449 \]

Oscillatory convergence to a limit cycle

At \( \lambda = 1 + \sqrt{6} \approx 3.449 \), the limit cycle gives way to a four cycle and at a little larger value of \( \lambda \), to an 8 cycle, and so on ad infinitum.

Stability Diagram

\[ ay'' + by' + cy = f(x) \quad a, b, c \text{ constant} \]

1) Find complementary functions, which satisfy the homogeneous equation

\[ ay'' + by' + cy = 0 \]

2) Find a particular integral that satisfies the full equation
Complementary functions

Recall that \( e^{ax} \) is an eigenfunction of \( \frac{d}{dx} \) and hence also \( \frac{d^2}{dx^2} = \frac{d}{dx} \frac{d}{dx} \). Therefore the complementary functions have the form:

\[
y_c = e^{ax}, \quad y_c' = xe^{ax}, \quad y_c'' = x^2 e^{ax}
\]

\[
\Rightarrow a^2 + b^2 + c = 0 \quad \text{characteristic equation}
\]

There are two (possibly complex) solutions of the characteristic equation. If they are distinct, \( \lambda_1, \lambda_2 \) say, \( \lambda_1 \neq \lambda_2 \), then there are two independent complementary functions:

\[
y_1 = e^{\lambda_1 x}, \quad y_2 = e^{\lambda_2 x}
\]

If \( \lambda_1, \lambda_2 \) are distinct, \( y_1, y_2 \) are linearly independent and complete. They form a basis of the space of solutions of the homogeneous equation.

The general complementary function is:

\[
y_c = Ae^{\lambda_1 x} + Be^{\lambda_2 x}
\]

**E.g.**

\[
y'' - 5y + 6y = 0 \quad \Rightarrow \lambda^2 - 5\lambda + 6 = 0
\]

\[
\Rightarrow (\lambda - 2)(\lambda - 3) = 0
\]

\[
y_c = Ae^{2x} + Be^{3x}
\]

\[
y'' + 4y = 0 \quad \Rightarrow \lambda^2 + 4 = 0
\]

\[
\lambda = \pm 2i \quad \Rightarrow \lambda_1 = e^{2ix}, \quad \lambda_2 = e^{-2ix}
\]

\[
y_c = A(\cos 2x + i\sin 2x) + B(\cos 2x - i\sin 2x)
\]

\[
y_c = (A + B)\cos 2x + i(A - B)\sin 2x
\]

\[
y_c = (A + B)\cos 2x + i(A - B)\sin 2x
\]
Differential Equations

Degeneracy:
\[ y'' - 4y' + 4y = 0 \]
\[ \lambda^2 - 4\lambda + 4 = 0 \]
\[ (\lambda - 2)^2 = 0 \]
\[ \lambda = 2 \text{ or } 2 \]
But \( e^{2x} \) and \( e^{-2x} \) are clearly not independent.
So these in particular are not complete.

Determining: Consider
\[ y'' - 4y' + (4 - \varepsilon^2)y = 0 \]
To find eigenfunction solution
\[ y = e^{\lambda x} \]
\[ \lambda^2 - 4\lambda + 4 - \varepsilon^2 = 0 \]
\[ \lambda = 2 \pm \frac{\varepsilon}{2} \]
\[ y_c = Ae^{(2 + \varepsilon)x} + Be^{(2 - \varepsilon)x} = e^{2x} \left( Ae^{\varepsilon x} + Be^{-\varepsilon x} \right) \]
\[ = e^{2x} \left[ (A + B) + \varepsilon x (A - B) + O(\varepsilon^2) \right] \]

Choose \( A + B = \alpha \), independent of \( \varepsilon \)
\[ \varepsilon (A - B) = \beta, \text{ independent of } \varepsilon \]
\[ A = \frac{1}{2} \left( \alpha + \frac{\beta}{\varepsilon} \right), \quad B = \frac{1}{2} \left( \alpha - \frac{\beta}{\varepsilon} \right) \]
\[ = O \left( \frac{1}{\varepsilon} \right) \]
\[ B = O \left( \frac{1}{\varepsilon} \right) \]
So \( \varepsilon \to 0 \)

\[ y_c = e^{2x} \left[ \alpha + \beta x + O(\varepsilon) \right] \]

Linear equations with constant coefficients
\[ \Rightarrow e^{2x} \left[ \alpha + \beta x \right], \quad \varepsilon \to 0 \]
A demonstration of a general rule that if \( y_1(x) \) is a degenerate complementary function, then \( y_2(x) = xy_1(x) \) is a complementary function.
Differential Equations

Method of finding second complementary functions (degenerate cases):

If \( y_1(x) \) is a complementary function of a homogeneous linear 2nd order ODE, look for another solution of the form \( y_2(x) = v(x) y_1(x) \).

Note that \( v'(x) \) will satisfy a first-order equation.

E.g. \( y'' - 4y' + 4y = 0 \), \( y_1 = e^{2x} \)

Try \( y_2 = v(x)e^{2x} \)

\[ y_2' = (v' + 2v)e^{2x} \]
\[ y_2'' = (v'' + 4v' + 4v)e^{2x} \]

\[ y'' + 4y' + 4y - 4(v' + 2v) + 4v = 0 \]

\( v'' = 0 \), \( v' = A \), \( v = Ax + B \)

So \( y_2(x) = (Ax + B)e^{2x} \)

Note that \( y_2 \) may include arbitrary amount of \( y_1 \).

This method works for any linear homogeneous ODEs, constant coefficient.

Phase Space: A differential equation of \( n^{th} \) order determines the \( n^{th} \) derivative \( y^{(n)}(x) \), and hence, all other derivatives in terms of \( x, y(x), y'(x), \ldots, y^{(n-1)}(x) \).

We can think of this in terms of a solution vector:

\[ Y = \begin{pmatrix} y(x) \\ y'(x) \\ \vdots \\ y^{(n-1)}(x) \end{pmatrix} \]

Defining a point (for each value of \( x \)) in an \( n \)-dimensional phase space. \( Y(x) \) traces out a trajectory in phase space.

\( y'' + 4y = 0 \)

\[ y_1 = e^{2x}, \quad y_2 = \sin 2x \]

\[ y_1' = 2e^{2x}, \quad y_2' = \cos 2x \]

The solution vectors are

\[ Y_1 = \begin{pmatrix} e^{2x} \\ 2e^{2x} \end{pmatrix}, \quad Y_2 = \begin{pmatrix} \sin 2x \\ \cos 2x \end{pmatrix} \]
The solutions $y_1(x)$ and $y_2(x)$ are independent solutions of the differential equation because if the vectors $y_1$ and $y_2$ are linearly independent i.e. if the Wronskian determinant:

$$W(x) = \begin{vmatrix}
y_1 & y_2 & y_3 & \ldots & y_n \\
y'_1 & y'_2 & y'_3 & \ldots & y'_n \\
y''_1 & y''_2 & y''_3 & \ldots & y''_n \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
 y^{(n-2)}_1 & y^{(n-2)}_2 & y^{(n-2)}_3 & \ldots & y^{(n-2)}_n
\end{vmatrix} \neq 0$$

For a 2nd order equation $W(x) = \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix}$

E.g. $W = \begin{vmatrix} \cos 2x & \sin 2x \\ -2 \sin 2x & 2 \cos 2x \end{vmatrix} = 2 \neq 0$

Or

$$W = \begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & (1+2x)e^{2x} \end{vmatrix} = e^{4x} (1+2x-2) = e^{4x} \neq 0$$

**Abel's Theorem**

With equation in standard form $y'' + p(x) y' + q(x) y = 0$

If $p, q$ are continuous then either $W \equiv 0$ or $W \neq 0$ for any value of $x$.
Differential Equations

Suppose $y_1$ and $y_2$ are two solutions. Then $y_2(y_1'' + p y_1' + q y_1) = 0$ and $y_1(y_2'' + p y_2' + q y_2) = 0$. Subtract to get $(y_2 y_1'' - y_1 y_2'') + p(y_2 y_1' - y_1 y_2') = 0$.

\[- W' - pW = 0 \]
\[
\Rightarrow W' + pW = 0
\]
\[
\Rightarrow W = W_0 e^{-\int p \, dx}
\]

The exponential is never zero so $W_0 = 0$ or $W \neq 0$ for any $x$.

Note: Any linear second order differential equation can be written in the form $y'' + A(x)y = 0$.

It can be shown that $W' + T(A)W = 0$, $W = W_0 e^{-\int T(A) \, dx}$ and Abel's Theorem holds.
Differential Equations

Partial Integrals
Method 1 - Guesswork

\( f(x) \)  \( \quad y_p(x) \)  \( A e^{mx} \)

\( e^{mx} \)
\( \sin kx \)
\( \cos kx \)
\( x^n \)
\( q_n(x) = a_n x^n + \ldots + a_1 x + a_0 \)

Remember that equation is linear, so we can superpose solutions corresponding to different forcings.

E.g., \( y'' - 5y' + 6y = 2x + e^{4x} \)
\( y_p = \alpha x + \beta + ce^{4x} \)
\( y_p' = \alpha + 4ce^{4x} \)
\( y_p'' = 16ce^{4x} \)
\( 16ce^{4x} - 5(\alpha + 4ce^{4x}) + 6(\alpha x + \beta + ce^{4x}) = 2x + e^{4x} \)
\( 16c - 20\alpha + 6\beta = 1 \Rightarrow c = \frac{1}{10} \)
\( -5\alpha + 6\beta = 0 \Rightarrow \beta = \frac{6}{10} = \frac{3}{5} \)
\( 6\alpha = 2 \Rightarrow \alpha = \frac{1}{3} \)

General solution: \( y = Ae^{2x} + \frac{3}{2}e^{4x} + \frac{3x}{10} + \frac{3}{5} \)

Note!

Can apply boundary conditions with only the complete solution \( y = y_c + y_p \)

Resonance Consider \( y + \omega_0^2 y = 0 \) \( \sin \omega t \) \( y_c = A \sin \omega t + B \cos \omega t \)

Here the forcing is linearly dependent on the eigenfunctions of the homogeneous ODE (i.e., on the complementary functions).

\( y_p = C \sin \omega t + D \cos \omega t \) will give \( y_p' + \omega_0^2 y_p = 0 \) so we cannot force.

This example is a simple harmonic oscillator forced at its natural (resonant) frequency.
Consider \( y + \omega_0^2 y = \sin \omega t \) \( \omega \neq \omega_0 \)

\[
\begin{align*}
\dot{y}_p &= c(\sin \omega t - \sin \omega_0 t) \\
\ddot{y}_p &= c(-\omega^2 \sin \omega t + \omega_0^2 \sin \omega_0 t)
\end{align*}
\]

no cosine needed, no need to take account of first derivative

Substitute

\[
\Rightarrow c(\omega_0^2 - \omega^2) = 1
\]

\[
\Rightarrow y_p = \frac{\sin \omega t - \sin \omega_0 t}{\omega_0^2 - \omega^2}
\]

\[
\omega_0 - \omega = \Delta \omega
\]

\[
y_p = -\frac{2}{(\omega + \omega_0) \Delta \omega} \sin \left( \frac{\omega_0 - \omega}{2} t \right) \sin \left( \frac{\omega_0 + \omega}{2} t \right)
\]

If the forcing is at a frequency close to the natural frequency we get beating, and as \( \Delta \omega \to 0 \), the envelope tends to \( \infty \) and we see initial linear growth.

Mathematically:

\[
\Delta \omega \to 0 \\
y_p = -\frac{2}{\omega + \omega_0} \cos(\omega_0 t) \times \left( \frac{t}{2} \right)
\]

General rule: If forcing is a linear combination of complementary functions, the particular integral has an amplitude proportional to \( t \) times the non-resonant guess (relates to ODEs with constant coefficients).
Method 2
Let $y_1(x)$, $y_2(x)$ be linearly independent functions of the ODE.

\[ y'' + p(x) y' + q(x) y = f(x) \]

The solution vector $X_1 = (y_1)$, and $X_2 = (y_2)$ form a basis of the phase space (solution space).

We can write

\[ Y_p(x) = u(x) X_1(x) + v(x) X_2(x) \]

Then

\[ y_p'' = u y_1'' + u' y_1' + v y_2'' + v' y_2' \]

Apply product rule

\[ y_p' = u y_1' + u' y_1 + v y_2' + v' y_2 \]

Compare with \( \#2 \Rightarrow y_1 u' + y_2 v'' = 0 \)
\[ y'' + py' + qy = f \]

\[ y_p = uy_1 + vy_2 \]

\[ y_p' = uy_1' + vy_2' \]

\[ \Rightarrow y_1u + y_2v' = 0 \]

\[ y_p'' = uy_1'' + u'y_1 + vy_2'' + v'y_2 \]
Differential Equations

\[ Y_1(x) = \begin{pmatrix} y_1' \\ y_1 \\ \end{pmatrix} \quad Y_2(x) = \begin{pmatrix} y_2' \\ y_2 \\ \end{pmatrix} \]

\[ Y_p = u(x)Y_1(x) + v(x)Y_2(x) \]

\[ y_p = u y_1' + v y_2' \]

\[ y_p' = u y_1'' + v y_2'' \]

\[ y_1 u'' + y_2 v' = 0 \]

Differential Equation

\[ y'' + p(x) y' + q(x)y = f(x) \]

\[ y_p'' = u y_1'' + u y_1' + v y_2'' + v y_2' \]

Substitute differential equation

\[ \Rightarrow y_1 u' + y_2 v' = f(x) \]

(1) and (2) give

\[ \begin{pmatrix} y_1' \\ y_2' \end{pmatrix} \begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} 0 \\ f \end{pmatrix} \]

\[ \begin{pmatrix} u' \\ v' \end{pmatrix} = \frac{1}{w} \begin{pmatrix} y_2' \\ -y_2 \end{pmatrix} \begin{pmatrix} 0 \\ f \end{pmatrix} \]

So solution exists providing \( W \neq 0 \)

\[ \Rightarrow u' = -\frac{y_2}{w} f \quad v' = \frac{y_1}{w} f \]
Eqn. \( y'' + 4y = \sin 2x \)

\[ y_1 = \sin 2x \quad \text{and} \quad W = -2 \]
\[ y_2 = \cos 2x \]

\[ y_p = u \sin 2x + v \cos 2x \]
\[ y_p' = u \cos 2x + v ( -2 \sin 2x ) \]
\[ y_p'' = -2u \sin 2x + 2v \cos 2x \]

Sub and solve to find \( u' = \frac{\cos 2x \sin 2x}{2} \), \( v' = -\frac{\sin 2x}{2} \)

\[ u = -\frac{1}{16} \cos 4x, \quad v = \frac{1}{16} \sin 4x - \frac{x}{4} \]

\[ y_p = \frac{1}{16} \left( - \cos 4x \sin 2x + \sin 4x \cos 2x \right) - \frac{x}{4} \cos 2x \]

\[ \frac{1}{16} \sin 2x - \frac{1}{4} x \cos 2x \rightarrow \text{found earlier by 'detuning'} \]

Piece of complimentary function

**Homogeneous Equations** (linear equidimensional equation)

\[ ax^2 y'' + bxy' + cy = f(x) \]

with \( a, b, c \) constants.

**Complementary functions.**

Note \( y = x^n \) is an eigenvector of the operator \( x \frac{d}{dx} \)

1. **To solve** \( ax^2 y'' + bxy' + cy = 0 \)

   - \( y = x^k \), \( y' = kx^{k-1} \), \( y'' = k(k-1)x^{k-2} \)
   - \( ak(k-1) + bk + c = 0 \) \( \Rightarrow k = k_1, k_2 \)
   - \( y_c = Ax^{k_1} + Bx^{k_2} \)

2. **Write** \( z = \ln x \), show \( a \frac{d^2 y}{dz^2} + (b-a) \frac{dy}{dz} + cy = f(e^z) \)

So this transformation converts an equidimensional equation into one with constant coefficients.
Characteristic equation \( a \lambda^2 + (b-a) \lambda + c = 0 \)
\[ y_c = A e^{k_1 x} + B e^{k_2 x} \Rightarrow \text{same solution} \]

If roots of the characteristic equation are equal then \( y_c = e^{kx} \)
\[ y_c = x^k, \quad x^k \log x \]
And if there is a resonant forcing proportional to \( x^k \) or \( x^k \log x \) then there is a particular again with logarithmic growth; form \( x^k \log x \) or \( x^k \log x \)

**Difference Equations for discrete variables**

\[ a y_{n+2} + b y_{n+1} + c y_n = f_n \]
Solve in a similar way to differential equations by exploiting linearity and eigen functions.

**Difference operator**
\[ \Delta[y_n] = y_{n+1} - y_n \]
has eigenfunction \( y_n = k^n \)
because \( \Delta[k^n] = k^{n+1} - k^n = k^n (k-1) = k y_n \)

To solve the difference equation, first look for complimentary functions satisfying \( a y_{n+2} + b y_{n+1} + c y_n = 0 \)

Try \( y_n = k^n \)
\[ a k^{n+2} + b k^{n+1} + c k^n = 0 \]
\[ a k^2 + b k + c = 0 \]
\[ \Rightarrow k = k_1, k_2 \]

General complimentary function \( y_n^{(c)} = A k_1^n + B k_2^n \) \( \neq k_1 = k_2 \)

Particular integrals

\[ \begin{align*}
    y_n^{(p)} &= (A+Bn)k_1^n, & \text{if} \quad k_1 = k_2 \\
\end{align*} \]
Particular Integrals (difference equations)

\[ y^{(n)} + A_2^k \lambda^n + A_1^k + A_1^k + \ldots + C_n + D \]

\[ k_1, k_2, \ldots, k_n \]
Differential Equations

Difference operator $D[y_n] = y_{n+1}$ has an eigenfunction $y_n = k^n$

E.g. Fibonacci Sequence $y_n = y_{n-1} + y_{n-2}$, $y_0 = y_1 = 1$

$y_{n+2} - y_{n+1} - y_n = 0$

$D^2[y_n] - D[y_n] - y_n = 0$

$\Rightarrow k^2 - k - 1 = 0$

$k = \frac{1 \pm \sqrt{5}}{2}$, $\varphi_1, \varphi_2$

General solution $y_n = A\varphi_1^n + B\varphi_2^n$

Initial conditions $y_0 = 1 = A + B$

$y_1 = 1 = A\varphi_1 + B\varphi_2$

$\Rightarrow A = \frac{\varphi_1}{\sqrt{5}}$, $B = -\frac{\varphi_2}{\sqrt{5}}$

$\Rightarrow y_n = \frac{\varphi_1^{n+1} - \varphi_2^{n+1}}{\sqrt{5}}$

Transient and damping

In many physical systems there is some sort of restoring force and some damping. E.g. car suspension

Newton's second law

$M\ddot{x} = F - kx = \frac{dL}{dt}$

$\ddot{x} + \frac{k}{M}\dot{x} + \frac{k}{M}x = \frac{F(t)}{M}$

Write $t = \sqrt{\frac{M}{k}}T$

$\ddot{x} + 2\frac{k}{M}\dot{x} + \frac{k}{M}x = f(T)$

where $(\cdot)$ means $\frac{\Delta}{\Delta t}$, $K = \frac{L}{2J_k m}$

There is a single parameter $K$ determining the behaviour of the system.

Free (natural response) $f = 0$, $\ddot{x} + 2\frac{k}{M}\dot{x} + \frac{k}{M}x = 0$

$\Rightarrow x = e^{\lambda t} \Rightarrow \lambda^2 + 2\frac{k}{M}\lambda + 1 \Rightarrow \lambda = -\frac{k}{M} \pm \sqrt{\frac{k^2}{M^2} - 1} = \lambda_1, \lambda_2$
If we increase the damping (or decrease the mass or spring constant) the period increases and the decay decreases.

As $K > 1$, period $> \frac{\pi}{K}$

$K = 1$ (critically damped) $x = (A + B\gamma) e^{-K \gamma}$

Possible to get a large initial increase in amplitude before the eventual slow decay

In a forced system, the complementary function determines the early time transient response while the particular integral determines the long-term "asymptotic" response.
Differential Equations

Given: 
\[ \ddot{x} + 2k \dot{x} + x = \sin \omega t \quad k \neq 0 \]

By 
\[ x = C \sin \omega t + D \cos \omega t \]

for particular integral 
\[ C = 0, \quad D = \frac{1}{2k} \]

\[ x = A e^{i \omega t} + B e^{-i \omega t} - \frac{1}{2k} \cos \omega t \sim -\frac{1}{2k} \cos \omega t \]

because 
\[ \text{Re}(A e^{i \omega t}) = 0 \]

Note the forced response is out of phase with the forcing.
Differential equations

Impulses and point forces
Consider a ball bouncing on the ground
\[ \text{Force exerted on the ball is } F(t) \]

\[ T \]
\[ t_1 \]
\[ t_2 \]

Often don't know or wish to know details of \( F(t) \) but note that it only acts for a time of \( O(\varepsilon) \) much less than the total time scale of the system

It is mathematically to imagine the force acting instantaneously at \( t = T \), i.e. \( \varepsilon \to 0 \). Using Newton's 2nd Law:
\[ \begin{align*}
\frac{d^2 x}{d t^2} &= F(t) - mg \\
\int_{T-E}^{T+E} m \frac{d^2 x}{d t^2} dt &= \int_{T-E}^{T+E} F(t) dt - mg \int_{T-E}^{T+E} dt
\end{align*} \]

\[ \left[ m \frac{d x}{d t} \right]_{T-E}^{T+E} = I - 2mg \varepsilon \quad \text{where } I = \int_{T-E}^{T+E} F(t) dt \]

are under the curve

If contact time \( 2\varepsilon \) is very small then mathematically we neglect it and write

\[ \left[ m \frac{d x}{d t} \right]_{T} = I \]

Important: The only feature of \( F(t, \varepsilon) \) we are interested in is its integral.
Mathematically, we consider a family of functions \( D(t; \varepsilon) \) such that

\[ \lim_{\varepsilon \to 0} D(t; \varepsilon) = 0 \quad \text{for all } t = 0. \]
\[ \lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} D(t; \varepsilon) dt = 1 \]

E.g. \( D(t; \varepsilon) = \frac{1}{\sqrt{4\pi \varepsilon}} e^{-\frac{t^2}{4\varepsilon}} \)

as \( \varepsilon \to 0^+ \), \( D(0; \varepsilon) \to 0 \) so \( \lim_{\varepsilon \to 0} D(t; \varepsilon) \) is not a function, it is undefined.
Nonetheless we define the Dirac Delta Function by 
\[ \delta(x) = \lim_{\varepsilon \to 0} g(x; \varepsilon) \]
on the understanding that we can only use its integral properties:

\[ \int_{-\infty}^{\infty} g(x) \delta(x) \, dx = g(0) \]

\[ \lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} g(x) \Delta(x; \varepsilon) \, dx \]

Note, no formal proof here.

Provided \( g \) is continuous.

This gives us a convenient way of representing and making calculations involving impulses or point forces.

\[ \ddot{x} = -mg + \int \delta(t-x) \, dt \]

\[ x = x_0, \quad x = 0, \quad t = 0. \]

\[ \ast \]

In general
\[ \int_{a}^{b} g(x) \delta(x-c) \, dx = g(c) \]

\[ = 0 \]

if \( c < a, \ c > b \)

**Example: Point Force**

Solve \( y'' - y = 3 \delta(x - \frac{\pi}{2}); \quad y = 0 \) at \( x = 0, \pi \)

for \( 0 \leq x \leq \pi \)

---

1. \( 0 \leq x < \frac{\pi}{2} \)
   \[ y'' - y = 0 \]
   \( y = 0 \) at \( x = 0 \) \( \Rightarrow B = 0 \)

2. \( \frac{\pi}{2} < x \leq \pi \)
   \[ y'' - y = 0 \]
   \( y = 0 \) at \( x = \pi \) \( \Rightarrow D = 0 \)

3. \( x = \frac{\pi}{2} \)
   \( y \) is continuous \( \Rightarrow A = C \)

Integrate from \( \frac{\pi}{2} - \epsilon \) to \( \frac{\pi}{2} + \epsilon \).

\[ y \left[ \frac{\pi}{2} + \epsilon \right] = 3 \]

\[ y = -\frac{3 \sinh x}{2 \cosh \frac{\pi}{2}} \]

\[ 0 \leq x < \frac{\pi}{2} \]

\[ -\frac{3 \sinh (\pi - x)}{2 \cosh \frac{\pi}{2}} \]

\[ \frac{\pi}{2} < x \leq \pi \]
Differential Equations

\[ \delta(x) = 0, \ x \neq 0 \]
\[ \int_{-\infty}^{\infty} \delta(x) \, dx = 1 \]
\[ \int_{a}^{b} g(x) \delta(x-c) \, dx = \begin{cases} g(c) \text{ if } a < c < b \\ 0 \text{ if } c \leq a \text{ or } c \geq b \end{cases} \]
\[ ay'' + by' + cy = \delta(x-d) \]
Then \[ ay'' + by' + cy = 0 \text{ if } x > d, \ x < d \]
\[ \frac{d^2 y}{dx^2} + a \frac{dy}{dx} = 0 \]

Heaviside Step Function \( H(x) \)
\[ H(x) = \int_{-\infty}^{x} \delta(t) \, dt \]
\[ H(x) = \begin{cases} 0, & x < 0 \\ 1, & x > 0 \end{cases} \]
\[ H(0) \text{ is undefined.} \]

Can apply the Fundamental Theorem of Calculus gives
\[ \frac{dH}{dx} = \delta(x) \]
useful for matching problems
\[ V(H(t)) = 1R + \frac{Q}{C} = R \frac{dq}{dt} + \frac{Q}{C} \]
\[ \Rightarrow \frac{Q}{C} + R \frac{dQ}{dt} = \frac{V}{R} H(t) \]
Note, \( Q \) is continuous at \( t=0 \) but \( Q \) jumps by \( \frac{V}{R} \)

Series Solutions
Consider equations of the form \( p(x) y'' + q(x) y' + r(x) y = 0 \)
\( x = x_0 \) is an ordinary point of the DE if \( \frac{q}{p} \) and \( \frac{r}{p} \) have Taylor series at \( x_0 \) (i.e. infinitely differentiable at \( x_0 \)). Otherwise it is a singular point. If \( x_0 \) is a singular point, but the equation can be written in the form
\[ p(x) \left( x - x_0 \right)^n y'' + q(x) \left( x - x_0 \right)^{n-1} y' + r(x) y = 0 \]
where \( \frac{q}{p} \) and \( \frac{r}{p} \) have Taylor series about \( x_0 \), then \( x_0 \) is a regular singular point.

Examples
i) \( (x^2 - 2x) y'' - 2xy' + 2y = 0 \). \( x = 0 \) is an ordinary point,
\( x = \pm 1 \) are regular points

ii) \( \sin x y'' + \cos x y' + 2y = 0 \)
\( x = \pi \) is a regular point, all regular. All others points are ordinary.
iii) $(1+5x)y'' - 2xy' + 2y = 0$
$x = 0$ is an irregular singularity.

**Theorem**
If $x_0$ is an ordinary point then the equation has 2 linearly independent solutions of the form
$$y = \sum_{n=0}^{\infty} a_n (x-x_0)^n$$
convergent in some neighborhood of $x_0$.

If $x_0$ is a regular singular point then the equation has at least 1 solution of the form
$$y = \sum_{n=0}^{\infty} a_n (x-x_0)^{n+\alpha}$$
Frobenius Series, $a_0 \neq 0$

Eqn: $(1-x^2)y'' - 2xy' + 2y = 0$, $x = 0$, ordinary point.
We will find a series solution about $x = 0$,
Write
$$(1-x^2)x^2 y'' - 2x^2 xy' + 2x^2 y = 0$$

Try:
$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$\sum_{n=0}^{\infty} a_n \left[ (1-x^2)(n(n-1)-2x^2 n + 2x^2) \right] x^n = 0$$

Coefficient of $x^n$ gives a general recurrence relation.
$$n(n-1)a_n - [(n-2)(n-2-1) + 2(n-2) + 2]a_{n-2} = 0$$
$$n(n-1)a_n = n(n-3)a_{n-2}$$

If $n \geq 2$:
$$a_n = \frac{n-3}{n-1} a_{n-2}$$

After this is the end of the story,
$$a_n = \frac{n-3}{n-1} a_{n-2} = \frac{n-3}{n-1} \cdot \frac{n-5}{n-3} a_{n-4}$$

$$\Rightarrow a_{2k+1} = -\frac{1}{2k+1} a_0, \quad k \geq 1$$

$$y = a_0 \left[ 1 - \frac{x^2}{1} - \frac{x^4}{3} - \frac{x^6}{5} \cdots \right] + q, x$$
$$= a_0 \left[ 1 - \frac{x}{1} \ln \frac{1}{1-x} \right] + q, x$$
\[
(1-x^2) y'' - 2xy' + 2y = 0
\]
\[
p y'' - q y' + ry = 0
\]
If \( \frac{q}{p} \) have Taylor series at \( x = x_0 \), then \( x_0 \) is a regular point.

1. \( x = 1 \) is a regular singular point.

\[
(1-x^2) y'' - xy = 0
\]

Example:
\[
x = 0 \text{ is a regular singular point.}
\]

First write
\[
4x^2 y'' + 2(1-x^2)(xy') - x^2(y) = 0
\]

Try
\[
y = \sum_{n=0}^{\infty} a_n x^{n+\sigma}, \quad a_0 \neq 0
\]

\[
\sum_{n=0}^{\infty} a_n \left[ 4(n+2)(n+2-1) + 2(1-x^2)(n+2) - x^2 \right] x^{n+\sigma} = 0
\]

Coefficient of \( x^{n+\sigma} \) gives
\[
\left[ 4(n+2)(n+2-1) + 2(n+2) \right] a_n + a_{n-2} \left[ -2(n+2) - 1 \right] = 0
\]

\[
2(n+2)(2n+2-1) a_n = (2n+2-3) a_{n-2}
\]

The case \( n = 0 \) gives the indicial equation, which determines the index \( \sigma \)
\[
25(2n+1) a_0 = 0 \quad \text{but we decided } a_0 \neq 0 \quad \Rightarrow \quad \sigma = 0, \quad \sigma = \frac{1}{2}
\]

Try \( \sigma = 0 \)

\[
2n(2n-1) = (2n-3) a_{n-2}
\]

\( n = 0 \) \( \Rightarrow \) \( a_0 = 0 \) \( \Rightarrow \) \( a_0 \) is arbitrary

\( n > 0 \) \( \Rightarrow \) \( a_n = \frac{2n-3}{4n(n-1)} a_{n-2} \), note \( a_1 = 0 \) \( \Rightarrow a_0 = 0 \) if \( n \) odd,
\[
a_{2k} = \frac{4k^2 - 3}{4k^2 - 1} a_{2k-2}
\]

\[y = a_0 \left[ 1 + \frac{x^2}{4} + \frac{3}{8\times 4\times 6} x^4 + \ldots \right]
\]

\( \sigma = 1 \)
\[
\frac{(2n+1)(2n)}{2} a_n = (2n-2) a_{n-2}
\]

\( n = 0 \) \( \Rightarrow \) \( a_0 = 0 \) \( \Rightarrow \) \( a_0 \) is arbitrary (call it \( b_0 \))

\( n = 1 \)
\[
6a_1 = 0 \quad \Rightarrow \quad a_1 = 0
\]

\( n > 1 \)
\[
a_n = \frac{n-1}{n(n+1)} a_{n-2} \quad \Rightarrow \quad y = b_0 \left[ 1 + \frac{x^2}{8} + \frac{3}{256} x^4 + \ldots \right]
\]
Behaviour near $x_0$:

Indicial equation has two roots (for the 2nd order equation) say $\alpha_1, \alpha_2$

i) $\alpha_2 - \alpha_1$ is not an integer then there are two linearly independent Frobenius series solutions

\[ y_1 = (x - x_0)^{\alpha_1} \sum_{n=0}^{\infty} a_n (x - x_0)^n \]

The other solution is of the form $y_2 = \sum_{n=0}^{\infty} b_n (x - x_0)^{\alpha_2 - \alpha_1} \ln (x - x_0) y_1$

Example: $x^2 y'' - y = 0 \quad \beta = 1, \quad \lambda = -x \quad \rho = -x$

So $x = 0$ is a regular singular point

\[ \sum a_n x^{n+\sigma} \left[ (n+\sigma)(n+\sigma-1) - \alpha \right] = 0 \]

Coefficient of $x^{n+\sigma} \quad (n+\sigma)(n+\sigma-1) a_n = a_{n-1}$

$n = 0$ gives indicial equation $\Rightarrow \sigma (\sigma - 1) a_0 = 0$

$a_0 \neq 0 \Rightarrow \sigma = 0, 1$

$\sigma = 1 : \quad (n+1) a_n = a_{n-1}$

$n = 0 : \quad 0 = a_0 \Rightarrow a_0$ is arbitrary

$n > 0 : \quad a_n = \frac{a_{n-1}}{(n+1)(n+2)} a_0$

\[ y_1 = a_0 x \left[ 1 + \frac{x}{2} + \frac{x^2}{12} + \frac{x^3}{144} + \cdots \right] \]

$\sigma = 0 : \quad n(n-1) a_n = a_{n-1}$

$n = 0 : \quad 0 = a_0 \Rightarrow a_0$ is arbitrary

$n = 1 : \quad 0 = a_1 \Rightarrow a_0 = 0 \Rightarrow a_0$ is arbitrary

But we chose $a_0 \neq 0$.

Suppose we allow $a_0 = 0$. Then $0 = a_1 \Rightarrow a_1$ is arbitrary.

$n > 1, \quad a_n = \frac{1}{n(n-1)} a_{n-1} = \frac{1}{n(n-1)(n-2)} a_1$

\[ y_2 = a_1 \left[ 1 + \frac{x}{2} + \frac{x^2}{12} + \frac{x^3}{144} + \cdots \right] = y_1, \text{which is the solution we found already.} \]

The other independent solution is actually

\[ y_2 = y_1 \ln x + \sum_{n=0}^{\infty} b_n x^n \]
Consider an infinitesimal displacement $\Delta s = (\Delta x, \Delta y)$. The change in $f(x, y)$ during this displacement is
$$df = \Delta f = \frac{\partial f}{\partial x} \Delta x + \frac{\partial f}{\partial y} \Delta y.$$ 

The gradient vector $\nabla f$ is defined by $\Delta f = \nabla f \cdot \Delta s$, where $\nabla f$ has Cartesian components $(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$. $\nabla f$ is the gradient of $f$. 

Write $ds = ds \hat{z}$, where $\hat{z} = 1$. Then
$$df = ds \nabla f \cdot \hat{z}.$$ 

$\frac{df}{ds}$ is the directional derivative of $f$ in the direction of $\hat{z}$. 

The gradient vector $\nabla f$ has the following properties: 

1. $\nabla f$ has magnitude equal to the maximum rate of change of $f(x, y)$ with distance in the $x$-$y$ plane. 
2. $\nabla f$ has direction in which $f$ increases most rapidly. 
3. If $\Delta s$ is a displacement along a contour of $f$, then $\frac{df}{ds} = 0$. 
4. $\nabla f$ is orthogonal to the contour. 

Examples of gradient vectors: 

- If $\Phi$ is the gravitational potential, $E = -\nabla \Phi$ is the gravitational force. 
- $T(x, y, z)$ is temperature, then heat flows by conduction in the direction of $-\nabla T$, so heat flow $q = -k \nabla T$ (thermal conductivity). 

Stationary Points: 

There is always one direction in which $\frac{df}{ds} = 0$ namely parallel to a contour of $f$. Local maxima and minima have $\frac{df}{ds} = 0$ for all directions. 

In Cartesian this translates to $\frac{df}{dx} = 0$, $\frac{df}{dy} = 0$. 

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but \( \nabla f = 0 \) also at middle points (pinnacle points)

Note: contours are locally elliptical at maxima and minima, whereas they are locally hyperbolic at saddle points.

Note: contours cross only at middle points.

**Taylor Series for multi-variable functions**

Consider a finite displacement \( SS \) along a straight line in the \( x-y \) plane. Then \( SS \Delta \vec{a} \)

\[
\text{The Taylor series along the line is}
\]

\[
f(S) = f(S_0 + SS) = f(S_0) + SS \frac{df}{ds} + \frac{1}{2} SS^2 \frac{d^2f}{ds^2} + \ldots
\]

\[
= f(S_0) + SS \cdot \nabla f + \frac{1}{2} \left( \left( \frac{\partial f}{\partial x} \Delta x + \frac{\partial f}{\partial y} \Delta y \right) \right) f + \ldots
\]

where \( SS \cdot \nabla f = SS \left( \frac{\partial f}{\partial x} \Delta x + \frac{\partial f}{\partial y} \Delta y \right) \)

\[
SS^2 \left( \frac{\partial^2 f}{\partial x^2} \Delta x^2 + \frac{\partial^2 f}{\partial y^2} \Delta y^2 \right) f
\]

\[
= SS^2 \left( \frac{\partial f}{\partial x} \Delta x + \frac{\partial f}{\partial y} \Delta y \right) \left( \frac{\partial f}{\partial x} \Delta x + \frac{\partial f}{\partial y} \Delta y \right)
\]

\[
= SS^2 f_{xx} + SS^2 f_{xy} + SSf_{yx} + SS^2 f_{yy}
\]

\[
(\nabla \Delta f) = \left( \frac{\partial^2 f}{\partial x^2} \Delta x^2 + \frac{\partial^2 f}{\partial y^2} \Delta y^2 \right)
\]

where \( \nabla \Delta f = \left( \begin{array}{c} f_{xx} \\ f_{xy} \\ f_{yx} \\ f_{yy} \end{array} \right) \) is called the **Hessian Matrix**.
**Differential Equations**

**System of Linear Equations**
Consider two dependent variables, \( y_1(t) \) and \( y_2(t) \)

\[
\begin{align*}
y_1' &= ay_1 + by_2 + f_1(t) \\
y_2' &= cy_1 + dy_2 + f_2(t)
\end{align*}
\]

Equivalence to higher order equations

\[
\begin{align*}
y_1'' &= ay_1 + by_2 + f_1(t) \\
y_2'' &= cy_1 + dy_2 + f_2(t)
\end{align*}
\]

\[
\begin{align*}
y_1' &= ay_1 + by_2 + f_1(t) \\
y_2' &= cy_1 + dy_2 + f_2(t)
\end{align*}
\]

Conversely

\[
\begin{align*}
y' &= Ay + By \\
y_1 &= y_1 \\
y_2 &= y_2
\end{align*}
\]

Any \( n \)th order ODE is equivalent to a system of \( n \) first order ODEs

Consider \( \ddot{y} - M \dot{y} + E = 0 \), try complimentary function \( y_c = e^{\lambda t} \)

\[
E.g.
\begin{align*}
\begin{vmatrix}
-1 & 2 & 1 \\
4 & -2 & 1 \\
2 & 1 & 0
\end{vmatrix} &= 0 \\
\lambda^3 + 6\lambda^2 + 1\lambda - 2 &= 0, \\
\lambda &= 1, -2, 1
\end{align*}
\]

\[
\begin{align*}
\begin{vmatrix}
1 & 2 & 1 \\
-4 & 2 & 4 \\
0 & 1 & 2
\end{vmatrix} &= 0 \\
\begin{vmatrix}
1 & 2 \\
4 & 2 \\
1 & 0
\end{vmatrix} &= 0 \\
\begin{vmatrix}
1 & 2 \\
4 & 2 \\
0 & 1
\end{vmatrix} &= 0
\end{align*}
\]

Particular Integral

\[
\begin{align*}
\begin{vmatrix}
u_1 & e^t \\
u_2 & e^t
\end{vmatrix} &= \begin{vmatrix}
1 & e^t \\
-2 & e^t
\end{vmatrix} = (4)e^t \\
\Rightarrow \begin{vmatrix}
3 & 2 \\
-1 & 1
\end{vmatrix} &= (4) \\
\begin{vmatrix}
u_1 \\
u_2
\end{vmatrix} &= (4) \\
\begin{align*}
u_1 &= -\frac{1}{2} \\
u_2 &= \frac{3}{2}
\end{align*}
\end{align*}
\]

General solution: \( y = A(1)e^t + B(-1)e^{-t} \)
Other linear phase-plane portraits:

- General solution to $\dot{y} = M \dot{x}$ is $y = Ax \cdot e^{\lambda t} + Bx \cdot e^{\lambda t}$
- $\lambda_1, \lambda_2 \text{ real, } \lambda_1, \lambda_2 < 0$ gives a saddle, e.g., $1x, 1 \lambda_2$
- $\lambda_1, \lambda_2 \text{ real, } \lambda_1 \lambda_2 > 0$ are nodes
- $\lambda_1, \lambda_2 \text{ complex conjugates}$

$\Re(\lambda_1) < 0$ $\bullet$ stable spiral
$\Re(\lambda_1) > 0$ $\circ$ unstable spiral
$\Re(\lambda_1) = 0$ $\bigcirc$ center
Differential Equations

General Non-Linear ODEs

In general, a 2nd order ODE can be written
\[ x' = f(x, y, t) \]
\[ y' = g(x, y, t) \]

An autonomous system of equations can be written
\[ x' = f(x, y) \]
\[ y' = g(x, y) \]

If the independent variable does not appear explicitly.

An n-th order, non-autonomous system can be converted into an (n+1)th order autonomous system by treating the former independent variable as a dependent variable. E.g. write \( z = t \)
\[ x' = f(x, y, z) \]
\[ y' = g(x, y, z) \]
\[ z' = 1 = h(x, y, z) \]

Equilibrium (fixed points) for 2nd order autonomous systems

\[ \begin{align*}
  x' &= 0 \\
  y' &= 0
  \end{align*} \]
\[ \begin{align*}
  f(x_0, y_0) &= 0 \\
  g(x_0, y_0) &= 0
  \end{align*} \]

Solve simultaneously

Stability: Write \( x = x_0 + \alpha \)
\[ y = y_0 + \beta \]
Substitute to find
\[ \begin{align*}
  \alpha' &= f(x_0 + \alpha, y_0 + \beta) \\
  \beta' &= g(x_0 + \alpha, y_0 + \beta)
  \end{align*} \]

If \( \alpha' = 0 \) and \( \beta' = 0 \), \( (\alpha, \beta) = 0 \)

Example: Population dynamics: Predator-Prey

Prey: \( \begin{align*}
  \frac{dx}{dt} &= Ax - Bx^2 - Cxy \\
  \frac{dy}{dt} &= -By + Cxy
  \end{align*} \)

Predators: \( \begin{align*}
  \frac{dx}{dt} &= dx - 2xy - Cxy \\
  \frac{dy}{dt} &= -y + xy
  \end{align*} \)

Fixed point:
\[ x = 0 \Rightarrow x(1 - 2x - zy) = 0 \]
\[ \Rightarrow x = 0, \ y = \frac{1}{2} - x \]
\[ y = 0 \Rightarrow y(x - 1) = 0 \Rightarrow y = 0, \ x = 1 \]

Fixed points at \( (0, 0), \ (1, 0), \ (1, 3) \)
\[ \begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \]
Near (4,0) \( x = 4 + \alpha, \ y = \beta \)

\[
\begin{pmatrix}
\alpha \\
\beta
\end{pmatrix}
= \begin{pmatrix}
-8 & -3 \\
0 & 3
\end{pmatrix}
\begin{pmatrix}
\alpha \\
\beta
\end{pmatrix}
\]

\( \hat{A} = (4+\alpha)(8 - 8 - 2\alpha - 2\beta) = -8\alpha - 8\beta \)
\( \hat{B} = \beta(3 + \alpha) = 3\beta \)

Eigenvectors \( [\alpha, \beta] \)

Near (1,3) write \( x = 1 + \alpha, \ y = 3 + \beta \)

\[
\begin{pmatrix}
\alpha \\
\beta
\end{pmatrix}
= \begin{pmatrix}
-2 & -2 \\
3 & 0
\end{pmatrix}
\begin{pmatrix}
\alpha \\
\beta
\end{pmatrix}
\]

Characteristic polynomial \( \lambda^2 + 2\lambda + 6 = 0 \)
Partial Differential Equations - Hyperbolic (wave) equations

Order for \( y(x, t) \) \( \frac{\partial y}{\partial x} = c \frac{\partial y}{\partial t} \), \( \frac{\partial y}{\partial t} - c \frac{\partial y}{\partial x} = 0 \) inforced.

Recall that along a path \( x = x(t) \), \( \frac{dy}{dt} = \frac{\partial y}{\partial x} \frac{dx}{dt} + \frac{\partial y}{\partial t} \frac{dt}{dt} = \frac{\partial y}{\partial x} \frac{dx}{dt} + \frac{\partial y}{\partial t} = \frac{dx}{dt} + \frac{dy}{dx} \frac{dx}{dt} \).

Choose to travel along a particular path defined by \( \frac{dx}{dt} = -c \). Then along that path \( \frac{dy}{dt} = 0 \). This method converts the PDE into several ODEs. The path is defined by \( x = -ct + x_0 \), \( x + ct = x_0 \) (constant).

Along the path \( y = A \) (constant).

There is a function \( f(x_0) \) that determines the value of \( y \) on each path \( \Rightarrow y = f(x_0) = f(x + ct) \). This is the general solution of the partial differential equation.

Usually, initial conditions are given; e.g., \( \frac{\partial y}{\partial x} = c \frac{\partial y}{\partial x} \) with \( y(x, 0) = x^2 - 3 \)

\[ x^2 - 3 = f(x) \Rightarrow f(x + ct) = (x + ct)^2 - 3 \]

\[ t=0 \quad y = A \]

wave velocity = \(-c\)

Example: \( \frac{\partial^2 y}{\partial x^2} + 5 \frac{\partial^2 y}{\partial x \partial t} = e^{-t} \)

\( y(x, 0) = e^{-x^2} \)

The "characteristic equation" defining the paths or "characteristics" of the PDE is \( \frac{dx}{dt} = 5 \Rightarrow x = 5t + x_0 \莱 \Rightarrow x_0 = x_0 - 5t \)

Along these paths, \( \frac{dx}{dt} = e^{-t} \), \( y = A - e^{-x_0^2} \)

At \( t=0 \), \( y = A - 1 \), \( x_0 = x \), \( A - 1 = e^{-x_0^2} \), \( A = 1 + e^{-x_0^2} \)

\( \Rightarrow y = (1 + e^{-x_0^2}) - e^{-t} \)

\( \Rightarrow y = 1 + e^{-x_0^2 - t} \)

Second Order Wave equations \( \frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \) (mass x acceleration = curvature)

\( \frac{\partial^2 y}{\partial t^2} - c^2 \frac{\partial^2 y}{\partial x^2} = 0 \)

the coefficients are constant, \( c \), \( y = f(x + ct) \) and \( y = g(x - ct) \) is also a solution. The equation is linear so solutions can be superposed.

\( y = f(x + ct) + g(x - ct) \)
Exercise: Show that \( \frac{\partial^2 y}{\partial t^2} - c^2 \frac{\partial^2 y}{\partial x^2} = -4c^2 \frac{\partial^2 y}{\partial x \partial \beta} \quad \beta = x - ct \)

Hence \( \frac{\partial \beta}{\partial x} = 0 \), \( \frac{\partial y}{\partial \beta} = h(t) \)

\( \Rightarrow y = f(x) + g(\beta) = f(x + ct) + g(x - ct) \) \( (f' = h) \)

Example: \( y = \frac{1}{1 + x^2} \), \( \frac{\partial y}{\partial t} = 0 \) at \( t = 0 \)

\( y \to 0 \quad \Rightarrow \quad x \to \pm \infty \)

Therefore, at time \( t = 0 \) we have \( f(x) + g(x) = \frac{1}{1 + x^2} \)

\( cf'(x) - cg'(x) = 0 \Rightarrow f' = g' \)

\( \Rightarrow f = g + \text{constant} \)

\( \beta \neq 0 \) by applying \( y \to 0 \quad \Rightarrow \quad x \to \pm \infty \)

\( f = g = \frac{1}{1 + (x - ct)^2} \quad \Rightarrow \quad y = f(x + ct) + g(x - ct) \)

\( y = \frac{1}{1 + (x + ct)^2} + \frac{1}{1 + (x - ct)^2} \)

\( t = 0 \)
Differential Equations

Hyperbolic Equations
\[ \frac{\partial^2 y}{\partial t^2} - c^2 \frac{\partial^2 y}{\partial x^2} = F \]

Elliptic Equations

Parabolic Equation

\[ \frac{\partial T}{\partial t} = K \frac{\partial^2 T}{\partial x^2} \]

Note that heat flux \( \dot{Q} = \frac{\partial T}{\partial x} \)

A large heat flux heats up a small heat flux where \( T(x, t) \) is temperature and \( K \) is called diffusivity.

Example: An infinitely long bar heated at one end

\[ x = 0 \quad x \to \infty \]

Suppose \( T(x, 0) = 0 \)

There is a similarity solution of the differential equation where \( \eta = \frac{2\sqrt{RT}}{x} \)

Then \( \frac{\partial T}{\partial \eta} = \frac{\partial T}{\partial x} \frac{\partial x}{\partial \eta} = -4RT \frac{1}{3} \theta'(\eta) = -2 \frac{1}{x} \theta'(\eta) \)

\[ \frac{\partial T}{\partial x} = \frac{d\theta}{d\eta} \frac{\partial x}{\partial \eta} = \frac{1}{2RT} \theta''(\eta), \quad \frac{\partial^2 T}{\partial x^2} = \frac{1}{2RT} \theta''(\eta) = 4RT \theta''(\eta) \]

\[ = K \frac{\partial^2 T}{\partial x^2} \Rightarrow - \frac{\eta}{2c} \theta' = \frac{4RT}{2c} \theta'' \Rightarrow \theta'' + 2\eta \theta' = 0 \]

Solve with an integrating factor: \( e^{\int \frac{2c}{4RT} d\eta} = e^{\eta^2} \)

\[ \Rightarrow (e^{\eta^2} \theta')' = 0, \quad \theta' = Ae^{-\eta^2} \]

\[ \theta = A \int e^{t^2} dt + B, \quad \theta = a \text{erf } \eta + \beta \]

where \( \text{erf } \eta = \frac{2}{\sqrt{t}} \int_0^\eta e^{-u^2} du \geq 1 \) as \( \eta \to \infty \)

\[ \theta(0) = 1 \Rightarrow \beta = 1 \]

\[ \theta(x) \to 0 \Rightarrow \alpha = -1 \]

Corresponds to...
\[ \Theta = 1 - \operatorname{erf}(\eta) = \operatorname{erfc}(\eta) \]

\[ T = \operatorname{erfc}\left(\frac{x}{2\sqrt{\delta t}}\right) \]

The solutions at all times are similar, they have the same functional form but have a scale in the \( x \) direction that depends on \( t \). The decay length is proportional to \( \sqrt{\delta t} \).