Dynamics and Relativity

Stephen Siklos

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Contents

Dynamics web page

www.damtp.cam.ac.uk/user/stcs/dynamics.html

This web page has or will have:

- examples sheets, which I will also give out in lectures;
- any hand outs that I give out in lectures (including this one)
- notes, which I will not give out.

Note about notes

The notes on the web page are not 'lecture notes' or 'notes of the course'. They cover everything in the lectures, but in greater depth, and they cover more topics. You should use the notes to supplement your lecture notes (for example, if you think the I have left out too many steps in the algebra) or to read, in the style of the course, beyond the course (for example, some proofs — usually not very illuminating ones that are not part of the course, or extended discussions on the topics in the course).

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Dynamics and Relativity ①

Differentiating Vectors

The basic operations on vectors are addition and multiplication by a scalar. For vectors depending on a parameter, t (time or a spacial coordinate) we need the rules: $(\underline{u}+\underline{v})^\circ = \dot{\underline{u}} + \dot{\underline{v}}$ $\quad (\lambda\underline{u})^\circ = \dot{\lambda}\underline{u} + \lambda\dot{\underline{u}}$

[These follow from the definition of differentiation ~~by a vector~~ $\frac{d}{dt}\underline{u} = \lim_{h\to 0}\frac{1}{h}\left[\underline{u}(t+h) - \underline{u}(t)\right]$

Then if $\underline{e_i}$ are fixed vectors of a basis $\quad \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}^\circ = \begin{pmatrix} \dot{u}_1 \\ \dot{u}_2 \\ \dot{u}_3 \end{pmatrix}$

$\dot{\underline{u}} = (u_i\underline{e_i})^\circ = \dot{u}_i\underline{e_i}$

But if $\underline{e_i}$ are not fixed then $\quad \dot{\underline{u}} = \dot{u}_i\underline{e_i} + u_i\dot{\underline{e_i}}$

e.g in plane polar coordinates and axes

The position vector ~~$\underline{v}$~~ is $\underline{r} = r\underline{e_r}$, $\quad \dot{\underline{r}} = \dot{r}\underline{e_r} + r\dot{\underline{e_r}} = \dot{r}\underline{e_r} + r\dot{\theta}\underline{e_\theta}$

Differentiating vector products

We have $\quad (\underline{u}\cdot\underline{v})^\circ = \dot{\underline{u}}\cdot\underline{v} + \underline{u}\cdot\dot{\underline{v}}$, $\quad (\underline{u}\times\underline{v})^\circ = \dot{\underline{u}}\times\underline{v} + \underline{u}\times\dot{\underline{v}}$

Proof $\quad (\underline{u}\times\underline{v})^\circ_i \equiv (\underline{u}\times\underline{v})_i^\circ = (\epsilon_{ijk}u_jv_k)^\circ = \epsilon_{ijk}\dot{u}_jv_k + \epsilon_{ijk}u_j\dot{v}_k$

$= (\dot{\underline{u}}\times\underline{v})_i + (\underline{u}\times\dot{\underline{v}})_i$

Chapter ① Basic Concepts

1.1 Newton's Laws of Motion

1.1.1 Statement of Newton's Laws

N1 Every particle remains at rest or moves with constant velocity unless acted on by a force.

N2 Force = rate of change of momentum

N3 To every action, there is an equal and opposite reaction.

1.1.2 N1 and inertial frames

N1 can be regarded as a special case of N2 ($\underline{F} = \underline{0}$)

A more useful interpretation is that N1 defines the set of frames (axes in which N2 holds.

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Dynamics and Relativity (2)

Accordingly, we define an ~~interted~~ inertial frame to be a set of axes in which N1 holds and we modify N2:

$\underline{F} = \frac{d}{dt}(m\underline{v})$ in any inertial Frame.

1.1.4 N2 and varying mass

Newton's laws apply to particles and in Newtonian Dynamics (ND) particles have fixed mass. In this case $\underline{F} = m\underline{a}$. However, when we extend N2 to systems of particles, or Special Relativity (SR), m can vary and the momentum formulation is correct.

1.1.7 Absolute time

Newton assumed that time is absolute, i.e. that all observers agree a common time (up to constant scaling i.e. choice of units, and choice of origin). In SR, we use a different assumption.

1.1.8 Galilean Transformations (GT)

The coordinate transformations that preserve absolute time and N1 are called GTs. They relate inertial frames in Newtonian Dynamics. It can be shown that any GT can be written as a combination of:

translations $\begin{cases} t \mapsto t + t_0 \\ \underline{x} \mapsto \underline{x} + \underline{x}_0 \end{cases}$ where t_0 and $\underline{x}_0$ are fixed

rotations and reflections $\begin{cases} t \mapsto t \\ \underline{x} \mapsto R\underline{x} \end{cases}$ where $R^TR = I$, R is a 3x3 orthogonal matrix, constant

boost $\begin{cases} t \mapsto t \\ \underline{x} \mapsto \underline{x} + \underline{v}t \end{cases}$ where $\underline{v}$ is a constant vector

Ignoring translations, we can express any GT in matrix form.

$$\begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = \left(\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline v_1 & & & \\ v_2 & & R & \\ v_3 & & & \end{array}\right) \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}$$

We can use the properties of matrix multiplication to show that these form a group.

The 'interesting' transformation is the boost. If we use the rotation to align the x axis with $\underline{v}$, so that $\underline{v} = (v, 0, 0)$, then the transformation is $t' = t$, $x' = x + vt$, $y' = y$, $z' = z$.

1.2 Dimensional Analysis

1.2.1 Dimensions

In a physical theory, many variables have dimensions (e.g. length, time, charge etc). In ND, the basic dimensions are length [L], mass [M] and time [T], and all variables have dimensions that are combinations of L, M and T. $[\text{Speed}] = LT^{-1}$ $\quad [\text{Force}] = MLT^{-2}$

$[G, \text{N's constant of gravitation}] = M^{-1}L^{3}T^{-2}$

All equations must be dimensionally consistent, i.e. each term must have the same dimensions.

Example Suppose $[y] = L$. Then this equation $y = x^2 + e^{x}$ is inconsistent. We have $[y] = L \Rightarrow [x] = L^{\frac{1}{2}}$

but then $e^{x} = 1 + x + \frac{x^2}{2} + \dots \quad \Rightarrow$ inconsistent!

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1.2.3 Bridgman's Theorem

Theorem For any physical quantity Q, $[Q] = L^{\alpha}M^{\beta}T^{\gamma} \ldots\ldots$ for some numbers α, β and γ.

Proof Uses scaling laws of physical quantities. But note that if $[Q] = f(L, M, T)$ and f is analytic (has a Taylor series) at the origin, then

$$f(L, T, M) = \sum_{n=0}^{\infty} a_n L^n \quad , \ a_n = a_n(T, M)$$

which is dimensionally inconsistent unless $a_n = 0 \ \forall n$ except for one, as required. Note also that [force] is not analytic at $T = 0$.

1.2.4 Simple Pendulum

We will investigate the way frequency depends on the parameters in the problem.

(i) Assume that the only parameters in the problem are length (L), mass (m) and g (acceleration due to gravity).

The (angular) frequency is given by $\omega = f(L, m, g)$

By Bridgman's Theorem, $[\omega] = [f] = M^{\alpha}L^{\beta}T^{\gamma}$

Since there are three parameters and three dimensions, we argue

$c \longrightarrow$ $f = cL^a m^b g^d$

c a dimensionless constant

Compare dimensions: $T^{-1} = L^a M^b (LT^{-2})^d$

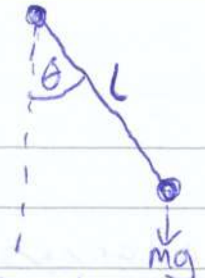
$\Rightarrow b = 0, \ d = \frac{1}{2}, \ a = -\frac{1}{2}$

$\omega = c\sqrt{\frac{g}{L}}$ (constant to be determined by solving equation of motion)

(ii) Now assume that ω in addition, depends on initial conditions, via the energy E, so $\omega = f(L, m, g, E)$

We can form one dimensionless parameter, $\lambda = \frac{E}{mgL}$ say

Now we can argue only that $f = g(\lambda) L^a m^b g^d$



We obtain $\omega = g(\lambda)\sqrt{\frac{g}{L}}$

To find $g(\lambda)$ we need to solve the equations of motions

$$\underbrace{\tfrac{1}{2}m(L\dot{\theta})^2}_{\text{kinetic energy}} \underbrace{- mgL\cos\theta}_{\text{potential energy relative to pivot}} = E$$

The quarter period $\frac{T}{4}$ is given by: $\frac{T}{4} = \sqrt{\frac{L}{2g}}\int_0^{\theta_0} \frac{d\theta}{\sqrt{\cos\theta - \cos\theta_0}}$

where $\cos\theta_0 = \frac{E}{mgL} = \lambda$

The integral is a function of θ_0 (θ is a dummy variable)

1.2.5 General method

In ND, if $n+3$ parameters $q_1, \ldots, q_{n+3}$ are given in the problem we can form n dimensionless parameters $\lambda_1, \ldots, \lambda_n$ and write any unknown quantity Q in the form $Q = f(\lambda_1, \ldots, \lambda_n)\, q_1^{\alpha} q_2^{\beta} q_3^{\gamma}$

(Buckingham Π Theorem)

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GI Taylor and the atomic bomb (1950)

Assume $R(t) = f(E, \rho, t)$

Fireball Radius; Explosion Energy; density of air; Time

So $E = C\rho^{\alpha} t^{\beta} R^{\gamma}$ $\qquad ML^2T^{-2} = (ML^{-3})^{\alpha} \cdot t^{\beta} L^{\gamma}$

$\Rightarrow E = C\rho t^{-2} R^5$

Chapter 2 Forces

2.1 Potentials

In 3D, a general force $\underline{F}$ is described by 3 independent functions, (F_1, F_2, F_3). In special cases, $\underline{F}$ can be written in terms of one function via force = - gradient of potential

2.1.1 Potentials in 1D

The work done by a force $F(x)$ in moving a particle from x to $(x+dx)$ is $F(x)dx$ and from x_0 to x is

Work done $= \int_{x_0}^{x} F(x')\,dx'$

The potential associated with $F(x)$ is defined up to an additive constant by $\phi(x) - \phi(x_0) = -\int_{x_0}^{x} F(x')\,dx'$

$\frac{d\phi}{dx} = -F(x)$ and Work Done $= -\phi(x) + \phi(x_0)$

2.1.2 Uniform Gravitational Field

$F(z) = -mg$ (z as height)

$\phi(z) - \phi(z_0) = -\int_{z_0}^{z} (-mg)\,dz'$

So $\phi(z) = mgz + \text{constant}$

2.1.3 Total Energy (E)

We define the energy of a particle of mass m in a potential $\phi(x)$ by $E = \frac{1}{2}m\dot{x}^2 + \phi(x)$ (2.3)

E is conserved: $\frac{dE}{dt} = m\dot{x}\ddot{x} + \frac{d\phi}{dx}\frac{dx}{dt}$

$\frac{dE}{dt} = \dot{x}F + (-F)\dot{x} = 0$ (using N2)

It is usually easier to use equation (2.3) than to use N2.

2.14 See handout

2.1.5 Potentials in 3D

The Work Done by a force $\underline{F}(\underline{r})$ in moving a particle from $\underline{r}$ to $\underline{r} + d\underline{r}$ is $\underline{F} \cdot d\underline{r}$ and from $\underline{r_0}$ to $\underline{r}$

$$WD = \int_{\underline{r_0}}^{\underline{r}} \underline{F}(\underline{r}') \cdot d\underline{r}'$$

which depends in general not just on $\underline{r}$ but also on the path, so it doesn't define a (unique) potential.

For some forces, WD is independent of the path (conservative) and we can define at potential $\phi(\underline{r})$ up to an additive constant, by

$$\phi(\underline{r}) - \phi(\underline{r_0}) = -\int_{\underline{r_0}}^{\underline{r}} \underline{F}(\underline{r}')\, d\underline{r}'$$

2·1·4

Hand-out 1: Motion in a cubic potential

A particle of unit mass moves in a one-dimensional potential $\phi(x)$, where

$$\phi(x) = x^3 - 3x\,.$$

The force due to this potential is $-\dfrac{d\phi}{dx}$ ('minus the gradient of the potential'), so the equation of motion of the particle is

$$\frac{d^2x}{dt^2} \equiv \ddot{x} = -\frac{d\phi}{dx} = -3x^2 + 3\,. \tag{1}$$

Multiplying by $\dfrac{dx}{dt}$ and integrating with respect to time gives the first integral (the energy integral)

$$\tfrac{1}{2}\dot{x}^2 = -\phi(x) + E$$

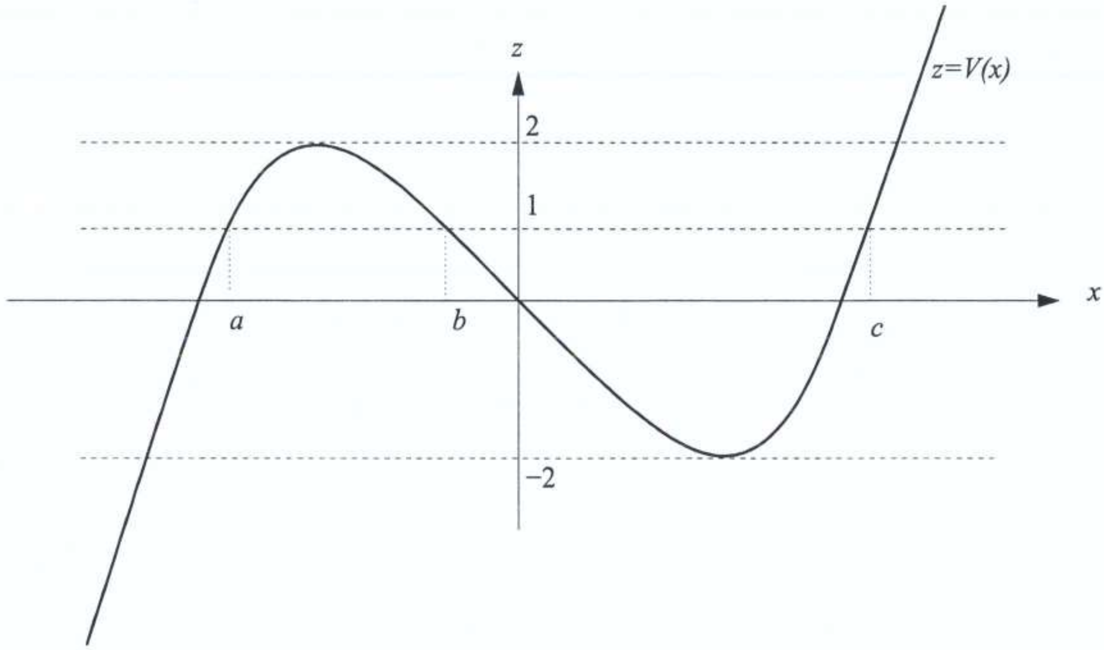
where E is a constant of integration (the total energy). This first order differental equation can also be integrated in principle to obtain

$$\int \frac{dx}{\sqrt{2E - 2(x^3 - 3x)}} = t.$$

This is an elliptic integral — it cannot be expressed in terms of elementary functions, though its properties have been well-studied.

A more illuminating approach comes from considering the equation of motion to be that of a particle of unit mass sliding under the action of gravity in a landscape the height of which above sea-level (say) is $\phi(x)$, as shown in the sketch. This approach works even for much more complicated potentials, where the integration approach would be unhelpful, and also for potentials that are functions of two variables.

The kinetic energy, and hence speed, of the particle is represented by the difference between the 'height' of the potential function and the fixed 'height' given by the total energy of the particle. At the points where these two heights coincide, the particle has zero speed but non-zero acceleration unless the point is a stationary point of the potential. For a smooth potential function, the particle will reverse when reaching such a point or, if it is a stationary point, will take an infinite amount of time to get there.



From the diagram, we can see the following possibilities (there are many others), depending on the initial conditions. For convenience, the initial conditions are given in terms of x_0 and E, rather than x_0 and $\dot{x}_0$.

(i) $x_0 < a$, $\dot{x}_0 > 0$, $E = 1$. In this case, the particle slows down until its velocity is reversed when $x = a$ (see diagram); it then goes off to $x = -\infty$.

(ii) $x_0 = a$, $E = 1$. The particle, initially stationary, sets off towards $-\infty$, gathering speed.

(iii) $a < x_0 < b$, $E = 1$. This is not possible: the particle does not have sufficient energy (classically) to exist in this part of the x-axis.

(iv) $b \le x_0 \le c$, $E = 1$. The particle oscillates between b and c.

(v) $x_0 > c$, $E = 1$. Again, not possible.

(vi) $E = 3$. The particle ends up at $-\infty$ either directly if $\dot{x}_0 \le 0$, or after bouncing off the potential if $\dot{x}_0 > 0$.

(vii) $E = 2$, $x_0 = -1$. Note that the turning points of $\phi(x)$ are at ± 1. In this case the particle has no kinetic energy and just stays put. It is in unstable equilibrium, as is obvious from the diagram. This can be checked analytically. Let $x = -1 + \epsilon$, where $\epsilon \ll 1$. Then, substituting into the equation of motion (1), we have

$$\frac{d^2}{dt^2}(-1 + \epsilon) = -3(-1 + \epsilon)^2 + 3 \approx +6\epsilon$$

so $\epsilon \approx \epsilon_0 \cosh \sqrt{6}(t - t_0)$, which grows grows exponentially. Small perturbations from the equilibrium will therefore in general become large, which means the equilibrium is unstable.

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$$\phi(\underline{r}) = \phi(\underline{r}_0) - \int_{\underline{r}_0}^{\underline{r}} \underline{F}(\underline{r}')\cdot d\underline{r}'$$

If we parametrise the path by t (path is $\underline{r}(t)$)

$$\phi(\underline{r}(t)) = \text{constant} - \int_{t_0}^{t} \underline{F}(\underline{r}(t'))\cdot \frac{d\underline{r}}{dt'}\,dt'$$

Differentiating with respect to t:

$\frac{d\phi}{dt} = -\underline{F}\cdot\frac{d\underline{r}}{dt}$ but $\frac{d\phi}{dt} = \frac{\partial\phi}{\partial x_i}\frac{dx_i}{dt}$ by the Chain rule

$\frac{d\phi}{dt} = \underline{\nabla}\phi \cdot \frac{d\underline{r}}{dt}$ so $\underline{F} = -\underline{\nabla}\phi$

Since $\frac{d\underline{r}}{dt}$ (which is tangent to the path) is arbitrary. (2.7)

Then we define the total energy E of the particle by $E = \frac{1}{2}m\dot{\underline{r}}\cdot\dot{\underline{r}} + \phi(\underline{r})$

in which case $\frac{dE}{dt} = m\dot{\underline{r}}\cdot\ddot{\underline{r}} + \frac{\partial\phi}{\partial x_i}\frac{dx_i}{dt} = \dot{\underline{r}}\cdot\underline{F} + (-\underline{F})\cdot\dot{\underline{r}} = 0$

In (2.7), $\phi(\underline{r})$ is called <u>potential energy</u> of the particle.

2.1.6 Central Forces

$\underline{F}(\underline{r})$ is said to be <u>central</u> if it can be written in the form

$\underline{F}(\underline{r}) = f(r)\hat{\underline{r}}$ where $r = |\underline{r}|$, $\hat{\underline{r}} = \frac{\underline{r}}{|\underline{r}|}$

~~Central Forces a~~ Central Forces are Conservative. $(\underline{\nabla} r = \hat{\underline{r}})$

Define $\phi(r)$ by $\frac{d\phi}{dr} = -f(r)$ so that $\underline{F} = -\frac{d\phi}{dr}\hat{\underline{r}} = -\underline{\nabla}\phi$

2.2 Friction

Two Types:

Dry friction (bodies in contact "$F = \mu R$")

Drag (body moving through a fluid)

$$r = (\underline{r}\cdot\underline{r})^{\frac{1}{2}}$$

$$\frac{dr}{dx_i} = \frac{1}{2}(\underline{r}\cdot\underline{r})^{-\frac{1}{2}}\left(2\underline{r}\cdot\frac{d\underline{r}}{dx_i}\right)$$

2·2·1 Drag

$k \propto$ viscosity of fluid × typical length scale

Drag is velocity dependent and is either linear $\underline{F} = -k\underline{v}$

(This applies to slowly moving particles such as rocks in lava)

or quadratic $\underline{F} = -k\,|\underline{v}|\,\underline{v}$ $\rightarrow k \propto$ density of the fluid × typical area scale

(which applies to faster motion such as projectiles in air.

2·2·3 See Handout

$$\cancel{\nabla(r) = \underline{e_i}\frac{\partial}{\partial x_i}(\underline{r}\cdot\underline{r})^{\frac{1}{2}} = \underline{e_i}\left(\frac{\partial r}{\partial x_i}\cdot\underline{r}\right)(\underline{r}\cdot\underline{r})^{-\frac{1}{2}}}$$

$$\underline{F}(\underline{r}) = f(r)\,\hat{\underline{r}}$$

$$\frac{d\phi}{dr} = -f(r)$$

$$\nabla\phi = \underline{e_i}\frac{\partial \phi}{\partial x_i} = \underline{e_i}\frac{d\phi}{dr}\frac{\partial r}{\partial x_i} = -f(r)\nabla r = -f(r)\,\hat{\underline{r}}$$

$$\nabla(r) = \underline{e_i}\frac{\partial}{\partial x_i}(x_j x_j)^{\frac{1}{2}} = \cancel{\underline{e_i}\frac{\partial x_j}{\partial x_i} r_j (r_k r_k)^{-\frac{1}{2}}}$$

$$= \underline{e_i}\frac{\partial x_j}{\partial x_i} x_j\, r^{-1} = \underline{e_i}\,\delta_{ij}\, x_j \frac{1}{r} = \frac{1}{r} x_i \underline{e_i}$$

$$= \frac{1}{r}\underline{r} = \hat{\underline{r}}$$

2-2-3

Hand-out 3: Projectile with linear drag

A particle of mass m is projected from the origin at velocity $\mathbf{u}$. The gravitational acceleration is denoted by $\mathbf{g}$ and the drag force is $-mk\mathbf{v}$, where k is a constant (the m is included here for convenience).

The equation of motion (Newton's second law) is

$$m\frac{d\mathbf{v}}{dt} = m\mathbf{g} - mk\mathbf{v} \quad \text{i.e.} \quad \frac{d\mathbf{v}}{dt} + k\mathbf{v} = \mathbf{g}.$$

We can solve this equation using an integrating factor, as if it were an ordinary (non-vector) differential equation. We first rewrite it as

$$\frac{d}{dt}\left(e^{kt}\mathbf{v}\right) = e^{kt}\mathbf{g}$$

then integrate and multiply by e^{-kt}:

$$\mathbf{v} = \frac{1}{k}\mathbf{g} + \mathbf{C}e^{-kt}$$

where $\mathbf{C}$ is a constant (vector) of integration which can be identified using the initial condition on the velocity which we take to be $\mathbf{v} = \mathbf{u}$ at $t = 0$. Thus

$$\mathbf{v} = \frac{1}{k}\mathbf{g} + (\mathbf{u} - \frac{1}{k}\mathbf{g})e^{-kt}.$$

This equation can be integrated directly to give $\mathbf{r}$:

$$\mathbf{r} = \frac{t}{k}\mathbf{g} - \frac{1}{k}(\mathbf{u} - \frac{1}{k}\mathbf{g})e^{-kt} + \mathbf{d}$$

where $\mathbf{d}$ is a (vector) constant of integration which can be identified using the initial condition on the position which we take to be $\mathbf{r} = 0$ at $t = 0$. Thus

$$\mathbf{r} = \frac{t}{k}\mathbf{g} - \frac{1}{k}(\mathbf{u} - \frac{1}{k}\mathbf{g})(e^{-kt} - 1). \qquad (*)$$

This is the complete solution. Choosing axes such that

$$\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad \mathbf{g} = \begin{pmatrix} 0 \\ 0 \\ -g \end{pmatrix} \quad \text{and} \quad \mathbf{u} = \begin{pmatrix} u\cos\alpha \\ 0 \\ u\sin\alpha \end{pmatrix},$$

the solution is

$$x = \frac{1}{k}u\cos\alpha\left(1 - e^{-kt}\right), \quad y = 0, \quad z = -\frac{gt}{k} + \frac{1}{k}\left(u\sin\alpha + \frac{g}{k}\right)\left(1 - e^{-kt}\right).$$

This looks a bit more complicated than the $k = 0$ case, but it is has some expected features. For very large t, in the sense $kt \gg 1$, the exponential terms can be ignored and (in this approximation) the particle drops vertically at its *terminal speed* of g/k; the horizontal component has been completely eroded by the frag force.

For small k (i.e. $kt \ll 1$), we should retrieve the projectile-without-drag solution. At first sight, this limit looks bad because of the k in the denominator. However, if we expand the exponential in the vector form of the solution $(*)$ as far as the quadratic terms we see that the limit is in fact defined (as it must be):

$$\begin{aligned} \mathbf{r} &= \frac{t}{k}\mathbf{g} - \frac{1}{k}(\mathbf{u} - \frac{1}{k}\mathbf{g})(1 - kt + \tfrac{1}{2}(kt^2) + \cdots - 1) \\ &= \mathbf{u}t + \tfrac{1}{2}\mathbf{g}t^2 + O(kt). \end{aligned}$$

This is the solution that we would have obtained by solving the equations of motion with $k = 0$.

02/02/11

Dynamics and Relativity 6

2-2-2 Particle Falling with Quadratic Drag

acceleration $m\frac{dv}{dt} = -mg + mkv^2$ included for convenience

We expect the particle to reach terminal velocity V_T where

$V_T = f(k, g, m)$. Using dimensional analysis

$f(k, g, m) = Ck^{\alpha}g^{\beta}m^{\delta}$

so $LT^{-1} = (L^{-1})^{\alpha}(LT^{-2})^{\beta}M^{\delta}$ $\quad \delta = 0,\ \beta = \frac{1}{2},\ \alpha = -\frac{1}{2}$

Thus $V_T = C\sqrt{\frac{g}{k}}$

To find C, we must solve the equation of motion.

$$-\int_0^v \frac{dv'}{g - kv'^2} = \int_0^t dt \quad , \quad -\frac{1}{\sqrt{gk}}\operatorname{artanh}\sqrt{\frac{k}{g}}v = t$$

$(-v \geq 0)$ $\Rightarrow -v = \sqrt{\frac{g}{k}}\tanh(\sqrt{gk}\,t)$

As $t \to \infty$, $v \to V_T$ which is equal to $\sqrt{\frac{g}{k}}$ i.e. $C = 1$.

Since $\tanh 1 \simeq 3/4$, $\frac{1}{\sqrt{gk}}$ is the time taken to reach $\frac{3}{4}V_T$

2.3 Motion in an Electromagnetic Field

2.3.1 Lorentz Force

The force on a charged particle moving in an E-M field is

$\underline{F} = e(\underline{E} + \underline{v} \times \underline{B})$ e - Charge on a particle, $\underline{E}$ - Electric Field, $\underline{v}$ - velocity, $\underline{B}$ - magnetic field

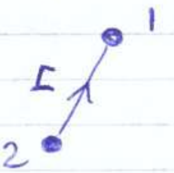
2.3.4 See Handout

2.4 Gravitational Forces

2.4.1 Newton's Law of Gravitation

The force between two particles is $\frac{Gm_1m_2}{d^2}$ (an attractive force)

In vector form $\underline{F}_{12} = -Gm_1m_2\frac{\underline{r}}{r^3}$



$\underline{F}_{12}$ is the force on particle 1 due to particle 2.

$\underline{F}_{12} = Gm_1m_2\underline{\nabla}\left(\frac{1}{r}\right)$

So the PE of 1 in the Gravitational Field of 2 is $-\frac{Gm_1m_2}{r}$

We define the gravitational potential of a point mass m at the origin by $\Phi(r) = -\frac{Gm}{r}$

Gravitational potentials are additive so for masses m_1 at $\underline{r}_1$ and m_2 at $\underline{r}_2$:

$$\Phi(\underline{r}) = -\frac{Gm_1}{|\underline{r}-\underline{r}_1|} - \frac{Gm_2}{|\underline{r}-\underline{r}_2|}$$

Mathematical Tripos Part IA
Dynamics and Relativity

Lent term 2011
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Hand-out 4: motion of a point charge in a uniform electromagnetic field

We wish to solve

$$m\ddot{\mathbf{r}} = e(\mathbf{E} + \dot{\mathbf{r}} \times \mathbf{B}) \qquad (*)$$

in the case when $\mathbf{E}$ and $\mathbf{B}$ are constant (in time) and uniform (same at all points in space).

The practical way to integrate the questions is to work in components, **BUT** it is essential to choose sensible axes. Since the lines of $\mathbf{B}$ are everywhere parallel, we can choose axes such that the z axis is parallel to $\mathbf{B}$:

$$\mathbf{B} = (0, 0, B)$$

If $\mathbf{E}.\mathbf{B} = \mathbf{0}$, we can choose axes such that $\mathbf{E} = (E, 0, 0)$, but in general the best we can do (by rotating the x and y axes, which is the only freedom left after fixing the z axis) is

$$\mathbf{E} = (E_1, 0, E_3).$$

With this choice, the equation of motion $(*)$ becomes

$$\begin{aligned} m\ddot{x} &= eE_1 + eB\dot{y} && (\dagger)\\ m\ddot{y} &= - eB\dot{x} && (\ddagger)\\ m\ddot{z} &= eE_3 \end{aligned}$$

which can be solved by elementary means or by using matrices.

The solution to third equation can be written down:

$$z = (e/2m)t^2E_3 + at + b$$

where a and b are constants obtainable from initial conditions.

A neat way to solve the $(\dagger)$ and $(\ddagger)$, which happens to work in this case, is to set $\xi = x+iy$, and add i times equation $(\ddagger)$ to equation $(\dagger)$; of course, one could always (for any pair of linear equations) do this to obtain a single complex equation containing both ξ and $\bar{\xi}$, but the special feature of our equations is that the result does not contain $\bar{\xi}$:

$$m\ddot{\xi} = eE_1 - ieB\dot{\xi}\,,$$

which can be integrated straight away:

$$\xi = pe^{-i\omega t} - iE_1t/B + q$$

where $\omega = eB/m$ and the complex constants p and q can be obtained from the initial conditions.[1]

If the particle is initially at the origin, and moving in the y-direction, we find

$$\xi = p(e^{-i\omega t} - 1) - ikt\,,$$

where $k = E_1/B$ and p is real, so

$$x = p(\cos\omega t - 1)\,, \qquad y = -p\sin\omega t - kt\,.$$

This is roughly (exactly if $k = p$) a cycloid, so the motion of the partical is, somewhat counter intuitively, a uniform acceleration parallel to $\mathbf{B}$ and cycloidal motion in the plane perpendicular to $\mathbf{B}$.

[1] ω is called the *Larmor frequency* after the physicist Joseph Larmor, senior wrangler in 1880, Lucasian Professor from 1903–1932.

04/02/

Dynamics and Relativity ⑦

3 Orbits

3.1 Motion in a Plane

3.1.1

For circular motion, it is convenient to use an angular variable θ instead of a linear variable x. The correspondence is as follows:

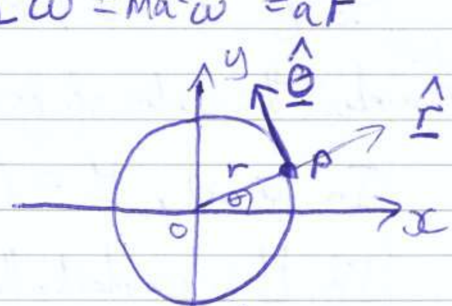
Motion in a straight line	Motion in a circle
Displacement x	Angular displacement θ
Velocity $\dot{x} = v$	Angular velocity $\dot{\theta} = \omega$
Acceleration $\ddot{x} = \dot{v}$	Angular acceleration $\ddot{\theta} = \dot{\omega}$
Kinetic energy $= \frac{1}{2}mv^2$	Kinetic energy $= \frac{1}{2}mv^2 = \frac{1}{2}ma^2\dot{\theta}^2 = \frac{1}{2}I\dot{\theta}^2$

a is the radius

where I is the 'moment of inertia' about the centre of the circle. $I = ma^2$

momentum mv	angular momentum $= I\omega = ma^2\omega$ $= a \times$ linear momentum
N2: $F = m\dot{v}$	N2: $\underset{\text{Torque}}{G} = I\dot{\omega} = ma^2\dot{\omega} = aF$

3.1.3 Acceleration in polar coordinates



In cartesians $\hat{\underline{r}} = (\cos\theta, \sin\theta)$

$\hat{\underline{\theta}} = (-\sin\theta, \cos\theta)$

so $\frac{\partial}{\partial r}\hat{\underline{r}} = \frac{\partial}{\partial r}\hat{\underline{\theta}} = 0 \qquad \frac{\partial \hat{\underline{r}}}{\partial \theta} = \hat{\underline{\theta}}, \quad \frac{\partial \hat{\underline{\theta}}}{\partial \theta} = -\hat{\underline{r}}$

In polar coordinates and axes we have $\underline{r} = r\hat{\underline{r}}$

$\Rightarrow \frac{d\underline{r}}{dt} = \frac{dr}{dt}\hat{\underline{r}} + r\frac{d\hat{\underline{r}}}{dt} = \dot{r}\hat{\underline{r}} + r\dot{\theta}\frac{\partial \hat{\underline{r}}}{\partial \theta}$

$\frac{d\underline{r}}{dt} = \dot{r}\hat{\underline{r}} + r\dot{\theta}\hat{\underline{\theta}}$

$\Rightarrow \frac{d^2\underline{r}}{dt^2} = (\ddot{r}\hat{\underline{r}} + r\dot{\theta}\hat{\underline{\theta}}) + (\dot{r}\dot{\theta}\hat{\underline{\theta}} + r\ddot{\theta}\hat{\underline{\theta}} - r\dot{\theta}^2\hat{\underline{r}})$

$\frac{d^2\underline{r}}{dt^2} = \ddot{\underline{r}} = (\ddot{r} - \underline{r\dot{\theta}^2})\hat{\underline{r}} + (r\ddot{\theta} + \underline{2\dot{\theta}\dot{r}})\hat{\underline{\theta}}$

centrifugal term   expected   coriolis term

$$\ddot{\underline{r}} = (\ddot{r} - r\dot{\theta}^2)\hat{\underline{r}} + \frac{1}{r}(r^2\dot{\theta})^{\cdot}\,\hat{\underline{\theta}}$$ Important Result

3.2 Angular Momentum

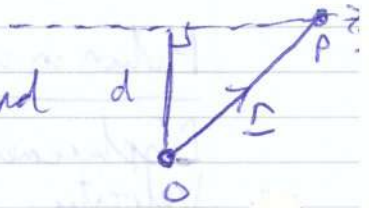
3.2.1 Definition

For a particle of mass m at $\underline{r}(t)$, we define the angular momentum about a fixed point $\underline{a}$ by ("moment of momentum")

moment of a general vector $\equiv \underline{r} \times \text{vector}$

$$\underline{H} = (\underline{r} - \underline{a}) \times (m\dot{\underline{r}})$$

If without loss of generality $|\underline{H}| = m|v|d$



where d is the shortest distance between the line of motion and the point.

3.2.2 Conservation of Angular Momentum

We have $\frac{d}{dt}\underline{H} = \frac{d}{dt}(\underline{r} - \underline{a}) \times (m\underline{v}) = \dot{\underline{r}} \times m\dot{\underline{r}} + (\underline{r} - \underline{a}) \times m\dot{\underline{v}}$

$\frac{d}{dt}\underline{H} = 0 + (\underline{r} - \underline{a}) \times \underline{F}$ $\quad \underline{G} = (\underline{r} - \underline{a}) \times \underline{F}$, torque about $\underline{a}$

So if $\underline{G} = 0$, $\underline{H}$ is conserved. If $\underline{F}$ is central so ($\underline{F}$ // to $\underline{r}$) then angular momentum about the centre of force is conserved.

In general, $\underline{H} = (\underline{r} - \underline{a}) \times \underline{p} \Rightarrow (\underline{r} - \underline{a}) \cdot \underline{H} = 0 \Rightarrow \underline{r} \cdot \underline{H} = \underline{a} \cdot \underline{H}$

If $\underline{H}$ is constant, this equation defines a plane with normal $\underline{H}$ through $\underline{a}$.

So for a central force, the motion lies in a plane.

07/02/11

Dynamics and Relativity 7

Acceleration in plane polars:

$$\ddot{\underline{r}} = (\ddot{r} - r\dot{\theta}^2)\hat{\underline{r}} + \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})\hat{\underline{\theta}}$$

Angular momentum about $\underline{a}$: $\underline{H} = (\underline{r} - \underline{a}) \times \underline{p}$

Conservation of angular momentum: $\frac{d}{dt}\underline{H} = (\underline{r} - \underline{a}) \times \underline{F}$ torque about $\underline{a}$

~~If F~~ If $\underline{F} \parallel \underline{r}$, torque about $\underline{O}$ is zero so angular momentum about the centre of force is conserved.

$\underline{H} \cdot \underline{r} = \underline{H} \cdot \underline{a}$ In this case, and without loss of generality, setting $\underline{a} = 0$, $\underline{H} = m\underline{r} \times \dot{\underline{r}} = m\underline{r} \times (\dot{r}\hat{\underline{r}} + r\dot{\theta}\hat{\underline{\theta}})$ in plane polars

$\underline{H} = mr^2\dot{\theta}\, \hat{\underline{r}} \times \hat{\underline{\theta}}$

So $|\underline{H}| = mr^2|\dot{\theta}|$ We define h, the angular momentum per unit mass of a particle moving in a plane by $h = r^2\dot{\theta}$

3.3 Orbits in a central force

Let $F(\underline{r}) = f(r)\hat{\underline{r}}$ (definition of central)

The angular momentum about $\underline{r} = 0$ is constant and $\underline{H} \cdot \underline{r} = 0$ defines a plane with normal $\underline{H}$, and the orbit lies in this plane

Using plane polar coordinates and vectors, the equation of motion is

$$(\ddot{r} - r\dot{\theta}^2)\hat{\underline{r}} + \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})\hat{\underline{\theta}} = \frac{1}{m}f(r)\hat{\underline{r}}$$

i.e. $\ddot{r} - r\dot{\theta}^2 = \frac{1}{m}f(r)$ $\quad r^2\dot{\theta} = h \leftarrow$ constant (3.13)

Eliminate $\dot{\theta}$ from the radial equation: $\ddot{r} = r\left(\frac{h}{r^2}\right)^2 = \frac{1}{m}f(r)$

$$\ddot{r} - \frac{h^2}{r^3} = \frac{1}{m}f(r) \qquad (3.15)$$

3·3·2 The $r(t)$ equation

We can integrate (3·1·5) once, $\frac{1}{2}\dot{r}^2 + \frac{h^2}{2r^2} = -\Phi(r) + A$

Where $\Phi'(r) = -\frac{1}{m}f(r)$ and A is a constant of integration.

Thus $\frac{1}{2}\dot{r}^2 + \underbrace{\frac{h^2}{2r^2} + \Phi(r)}_{\text{effective potential}} = A$ from which the motion can be understood using a graph of the effective potential. (2·1·4)

We can integrate once more, in principle

$$\pm\int\frac{dr}{\sqrt{2A - 2\Phi(r) - h^2/r^2}} = \int dt$$ to give $r(t)$ and then

$\theta(t)$ from (3·1·3), but this may not be helpful in practice.

3·3·3 The $u(\theta)$ equation

Instead we can change the dependent variable r to $u = \frac{1}{r}$ and change from t to θ to obtain a linear equation in u.

We have $\frac{d}{dt} = \dot{\theta}\frac{d}{d\theta} \overset{(3\cdot1\cdot3)}{=} hu^2\frac{d}{d\theta}$, $\frac{dr}{dt} = \frac{d(\frac{1}{u})}{dt} = hu^2\frac{d}{d\theta}\left(\frac{1}{u}\right) = -h\frac{du}{d\theta}$

and $\frac{d^2r}{dt^2} = \frac{d}{dt}\left(-h\frac{du}{d\theta}\right) = hu^2\frac{d}{d\theta}\left(-h\frac{du}{d\theta}\right) = -h^2u^2\frac{d^2u}{d\theta^2}$

Substitute into (3·1·5) $\quad -h^2u^2\frac{d^2u}{d\theta^2} - h^2u^3 = \frac{1}{m}f\left(\frac{1}{u}\right)$

$$\frac{d^2u}{d\theta^2} + u = -\frac{1}{mh^2u^2}f\left(\frac{1}{u}\right)$$

Exercise

$$KE = \frac{1}{2}\dot{\underline{r}}\cdot\dot{\underline{r}} = \frac{1}{2}h^2\left[\left(\frac{du}{d\theta}\right)^2 + u^2\right]$$

?9/02/11

Dynamics and Relativity (9)

3·5 Motion in an inverse square force

Let $f(r) = -mk/r^2$ (for convenience) where eg. $k = \begin{cases} GM & \text{Newtonian Gravity} \\ -\frac{qQ}{4\pi\varepsilon_0 m} & \text{Electrostatic force between point charges } q \text{ and } Q \end{cases}$

The Geometric Orbit equation becomes

$$\frac{d^2u}{d\theta^2} + u = \frac{k}{h^2} \qquad \left(-\frac{f(r)}{mh^2u^2}\right)$$

The general solution is $u = A\cos(\theta - \theta_0) + \frac{k}{h^2}$

We can choose axes such that $\theta_0 = 0$, then if $A < 0$ we set $\theta \to \theta + \pi$ so that the new A is positive. Thus

$$\boxed{u = A\cos\theta + \frac{k}{h^2}} \qquad (A \geq 0)$$

$\theta = 0$ corresponds to the maximum value of u and hence the minimum value of r: $\frac{1}{r_{min}} = A + \frac{k}{h^2}$

Setting $L = \frac{h^2}{|k|}$ and $e = \frac{Ah^2}{|k|}$

gives $u = \frac{1}{L}(e\cos\theta \pm 1)$, $r = \frac{L}{e\cos\theta \pm 1}$

where the + sign corresponds to $k > 0$, and the − to $k < 0$.

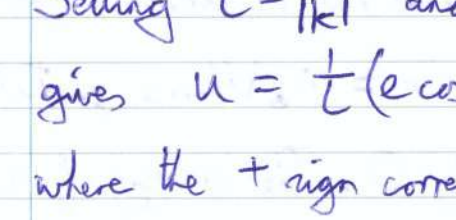
This is the polar equation of a general conic section.

I) $e = 0$, $k > 0$ a circle of radius L

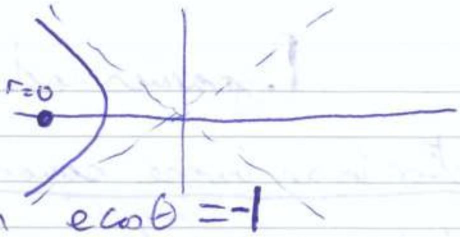
Check: $\frac{k}{L^2} = \frac{v^2}{L}$, $h = Lv \Rightarrow L = \frac{h^2}{k}$ as required

II) $e = 1$, $k > 0$ a parabola $r = \frac{L}{\cos\theta + 1}$, focus at $r = 0$

III) $0 < e < 1$, $k > 0$ Ellipse

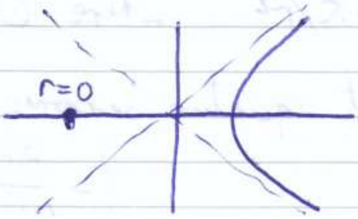


IV) $e > 1$, $k > 0$



Hyperbola $r_{max} = \infty$ when $e\cos\theta = -1$

V) $e > 1$, $k < 0$



Asymptote is $\theta = \theta_0$

Where $\cos\theta_0 = \frac{1}{e}$

Exercise:

Total energy $= m\frac{h^2}{2l^2}(e^2 - 1)$

$e \geq 1$

Note that $E \geq 0$ if, and the particle has enough energy to reach $r = \infty$ (an unbounded orbit)

$E < 0$ if $0 \leq e < 1$ and the particle cannot escape to $r = \infty$, so the orbit is bounded.

We define the escape "velocity" to be the speed that the particle requires at a given value of r in order to reach $r = \infty$. When a particle has the escape speed, its total energy is exactly 0. The energy of a particle moving on a parabola is zero, so the particle has the escape velocity at each point.

11/02/11

Dynamics and Relativity 10

Summary

The solution of $u'' + u = \frac{k}{h^2}$ is

$u = A\cos\theta + \frac{k}{h^2}$ $(A \geq 0)$

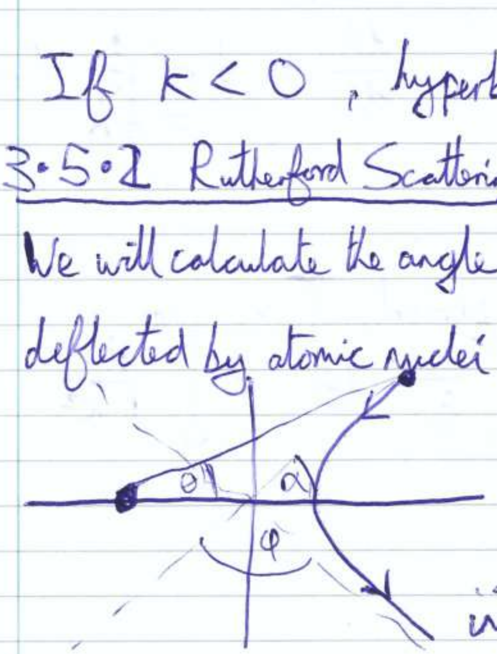
Orbits are $k > 0$ $\begin{cases} \text{circles } A = 0 \\ \text{ellipses } A < \frac{k}{h^2} \\ \text{parabolae } A = \frac{k}{h^2} \end{cases}$ hyperbolae $A > \frac{k}{h^2}$

If $k < 0$, hyperbolae, $A > \frac{k}{h^2}$

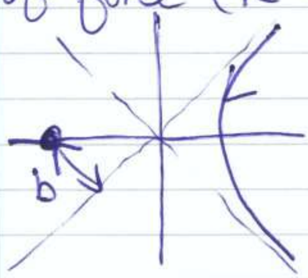
3·5·2 Rutherford Scattering

+vely charged

We will calculate the angle ϕ through which the alpha particles are deflected by atomic nuclei (also +vely charged). This is a hyperbola ($k<0$)



$u = A\cos\theta + \frac{k}{h^2}$. The unknown parameters A and h^2 can be expressed in terms of initial speed V (the speed at $u \simeq 0$) and impact parameter b which is the distance of closest approach in the absence of force ($k = 0$). We have $\underset{\text{(angular momentum)}}{h} = bV$



For large r, velocity is approximately radial. $\dot{\underline{r}} = (-V, 0)$ (polar coordinates)

$\equiv (\dot{r}, r\dot{\theta})$

$\equiv (-hu', hu)$, $u = \frac{1}{r}, h = r^2\dot{\theta}$

$= (+hA\sin\theta, h(A\cos\theta + \frac{k}{h^2}))$

As $r \to \infty$, i.e. $u \to 0$, $\theta \to \alpha$ where $A\cos\alpha = -\frac{k}{h^2}$

and $-V = hA\sin\alpha$. Eliminating A gives $\tan\alpha = -\frac{Vh}{k}$

and $\tan\frac{\phi}{2} = \frac{\tan(\pi - 2\alpha)}{2} = \cot\alpha = -\frac{k}{V^2 b}$

3.5.3 Keplers Laws (1605)

k1 Each planet moves on an ellipse with the sun at its focus.

k2 The radius vector sweeps out equal areas in equal times

k3 The orbital period is proportional to (mean orbital radius)$^{\frac{3}{2}}$

Proof: k1 is a consequence of the inverse square law.

k2  $\delta A = \frac{1}{2} r^2 \delta\theta$, $\frac{dA}{dt} = \frac{1}{2} r^2 \dot{\theta} = \frac{1}{2} h$ (constant)

This holds for any central force.

k3 For k3 we must calculate the period, T.

$$T = \int_{\text{one orbit}} dt = \int_0^{2\pi} \frac{dt}{d\theta}\, d\theta = \int_0^{2\pi} \frac{1}{\dot{\theta}}\, d\theta = \int_0^{2\pi} \frac{d\theta}{h u^2}$$

16/02/11 Dynamics and Relativity (11)

$$r_{min} = \frac{1}{A + \frac{k}{h^2}} \quad , \quad r_{max} = \frac{1}{-A + \frac{k}{h^2}}$$

$$\int_0^{2\pi} \frac{d\theta}{(a\cos\theta + b)^2} = \frac{2\pi b}{(b^2 - a^2)^{3/2}}$$

~~3.4 Stability~~ 3.4 Circular Orbits

3.4.1 Existence

A closed orbit has $r(\theta + 2\pi n) = r(\theta)$ for some $n \in \mathbb{Z}^+$

A special case is a circular orbit $r = r_0$

From $\ddot{r} - \frac{h^2}{r^3} = \frac{1}{m} f(r)$ we see that $-\frac{h^2}{r_0^3} = \frac{1}{m} f(r_0)$

There is a circular orbit of any radius r_0, provided $f(r_0) < 0$ (choose h)

3.4.2 Stability

We consider only radial perturbations since tangential perturbations correspond to $h \to h + \delta h$ with δh constant giving a stable perturbation

The radial equation is $\ddot{r} = g(r)$ where $g(r) = \frac{h^2}{r^3} + \frac{1}{m} f(r)$

Let $r = r_0 + \eta$, $\eta \ll r_0$

so $(r_0 + \eta)^{\cdot\cdot} = g(r_0 + \eta)$, $\ddot{\eta} = \eta g'(r_0) + \dots$

$$\ddot{\eta} = \eta \left[-\frac{3h^2}{r_0^4} + \frac{1}{m} f'(r_0) \right] + \dots$$

$$\ddot{\eta} = \eta \left[\frac{3f''(r_0)}{r_0} + f'(r_0) \right] \frac{1}{m} + \dots$$

The orbit is stable if $\frac{3f(r_0)}{r_0} + f'(r_0) < 0$

If $f(r)$ is a power law, $f(r) = -kr^n$, $k > 0$

$n >= 3 \Rightarrow$ stability

Chapter 4 - Rotating Frames

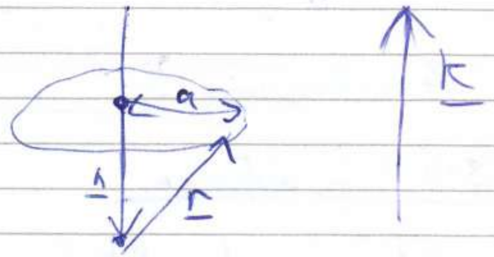
4.1 Angular velocity

For a particle rotating about a fixed axis $\parallel \underline{k}$ $(|k| = 1)$:

$\underline{r} = (a\cos\theta, a\sin\theta, h)$

$\dot{\underline{r}} = (-a\sin\theta, a\cos\theta, 0)\dot{\theta}$

$\quad = \underline{\omega} \times \underline{r}, \quad \underline{\omega} = \underline{k}\dot{\theta}$



$\underline{\omega}$ is the angular velocity, $\underline{\omega} \parallel \underline{k}$

4.2 Rotating Axes

We say that cartesian axes are rotating (with respect to a fixed inertial set) if $\dot{\underline{e}}_i = \underline{\omega} \times \underline{e}_i$ for some $\underline{\omega}$

14/02/11 Dynamics and Relativity (11)

$$T = \int_0^{2\pi} \frac{d\theta}{hu^2} = \frac{2\pi}{\sqrt{k}} \left(\frac{r_{min} + r_{max}}{2}\right)^{\frac{3}{2}} \leftarrow \text{mean radius}$$

$$\left[u = A\cos\theta + \frac{k}{h^2},\ r_{min} = \frac{1}{A+\frac{k}{h^2}},\ r_{max} = \frac{1}{-A+\frac{k}{h^2}} \right]$$

$$\left(\int \frac{d\theta}{(a\cos\theta + b)^2} = \frac{2\pi b}{(b-a)^{\frac{3}{2}}} \right)$$

3.4 Circular Orbits

3.4.1 Existence

A closed orbit has $r(\theta + 2\pi n) = r(\theta)$ for some $n \in \mathbb{Z}, n \geq 1$

A special case is a circular orbit, $r = r_0$

From $\ddot{r} - \frac{h^2}{r^3} = \frac{1}{m} f(r)$

We see that $-\frac{h^2}{r_0^3} = \frac{1}{m} f(r_0)$. There is a circular orbit of any radius r_0 provided $f(r_0) < 0$, we can choose h.

3.4.2 Stability

We consider only radial perturbations since tangential perturbations correspond to $h \to h + \delta h$ with δh constant, giving a stable perturbation (doesn't grow). The radial equation is $\ddot{r} = g(r)$ where

$g(r) = \frac{h^2}{r^3} + \frac{1}{m} f(r)$, $g(r_0) = 0$

Let $r = r_0 + \eta$ $\quad \eta \ll r_0$

so $(r_0 + \eta)^{\cdot\cdot} = g(r_0 + \eta)$, $\ddot{\eta} = \eta g'(r_0) + \dots$

$\ddot{\eta} = \eta \left[-\frac{3h^2}{r_0^4} + \frac{1}{m} f'(r_0) \right] + \dots = \eta \left[\frac{3f(r_0)}{r_0} + f'(r_0) \right] \frac{1}{m} + \dots$

The orbit is stable if $\frac{3f(r_0)}{r_0} + f'(r_0) < 0$

If $f(r)$ is a power law $f(r) = -kr^n$, $k > 0$

$n > -3 \Rightarrow$ stability

(NB $-2 > -3$)

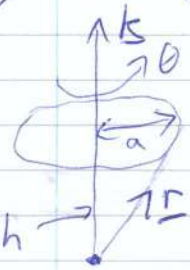
3.5 Non-Newtonian Orbits

See handout 5

Chapter 4 Rotating Reference Frames

4.1 Angular Velocity

For a particle rotating about a fixed axis $\parallel \underline{k}$ $\quad (|\underline{k}| = 1)$



$$\underline{r} = (a\cos\theta,\ a\sin\theta,\ h)$$

$$\dot{\underline{r}} = (-a\sin\theta,\ a\cos\theta,\ 0)\,\dot{\theta} = \underline{\omega} \times \underline{r}$$

$\underline{\omega} = \underline{k}\dot{\theta}$ where $\underline{\omega}$ is the angular velocity

(NB $\underline{\omega} \parallel \underline{k}$)

4.1.2 Rotating Axes

We say that Cartesian Axes $\underline{e_i}$ are rotating (wrt a fixed inertial set) if $\dot{\underline{e_i}} = \underline{\omega} \times \underline{e_i}$ for some $\underline{\omega}$

Mathematical Tripos Part IA
Dynamics and Relativity

Lent term 2011
Dr S.T.C. Siklos

Hand-out 5: Orbits in a non-inverse square force law.

In this example, we consider the following modification to Newtonian gravity:

$$f(r) = -\frac{k}{r^2} - \frac{a}{r^4},$$

where $k = GM$ as usual.

The $u(\theta)$ equation is

$$\frac{d^2u}{d\theta^2} + u = \frac{k}{h^2} + \frac{au^2}{h^2} \equiv \frac{1}{\ell}\left(1 + \lambda\ell^2u^2\right)$$

where $\ell = h^2/k$ and $\lambda = ak/h^4$. The reason for writing the right hand side in this form is that λ is a dimensionless parameter (note that ℓ has dimensions of length).

If we had chosen the additional term in the force to be proportional to r^{-3} instead of r^{-4}, we could have integrated the u-$d\theta$ equation; but, with the u^2 term on the right hand side, we cannot. Instead, we will obtain an approximation to the solution in the case $\lambda << 1$.

In the absence of the extra term (i.e. if $\lambda = 0$) we would obtain a solution corresponding to a Newtonian orbit, namely an ellipse. Our approximate solution will look very like an ellipse, but one that is slowly rotating in its own plane. We will calculate the rate of rotation.

We can therefore approximate the solution by iteration. The unperturbed solution is the Newtonian solution

$$u = \ell^{-1}(1 + e\cos\theta)$$

where $\ell = h^2/k$. We may verify that

$$u = \ell^{-1}(1 + e\cos((1-\lambda))\theta)$$

satisfies, approximately, the orbital equation [no need to do this — it is not the point of this example]. For the approximation, we use $\cos(\lambda\theta) \approx 1$ and $\sin\lambda\theta \approx \lambda\theta$ and ignore terms in λ^2.

At $r = r_{min}$ (the perihelion, for a planetary orbit), $\cos(1-\lambda)\theta = 1$. If the first is when $\theta = 0$, then the second is when $(1-\lambda)\theta = 2\pi$, i.e. when $\theta \approx 2\pi(1+\lambda)$. The approximate solution is therefore the ellipse corresponding to the unperturbed solution rotating slowly at the rate of $2\pi\lambda$ radians per orbit.

In fact, this modification to Newtonian gravity, with $a = 3GM/c^2$ (M is the mass of the sun), is the exact equation for a planetary orbit in General Relativity (a geodesic in the Schwarzschild solution).

For most astrophysical situations, the extra term is small. Nevertheless, planetary orbits have been observed for many centuries and even very small non-Newtonian affects are detectable.

Putting in the data for Mercury gives $\lambda \approx 10^{-7}$ and an advance of 43 arc second per century. Remarkably, it was known several decades before general relativity was formulated that out of a total observed precession of 5000 arc seconds per century, only 43 arc seconds are unexplained by Newtonian effects (such as the influence of other planets). The eccentricity of the orbit of Mercury is about 0.2 (compare with 0.016 for the Earth) and it is the least circular of any of the planetary orbits. Nevertheless, the accuracy of the observations is astounding.

16/02/11

Dynamics and Relativity (12)

$\dot{\underline{e_i}} = \underline{\omega} \times \underline{e_i}$

Here, $\dot{\underline{e_i}}$ is the velocity relative to non-rotating axes of a point a unit distance up the axis; and $\underline{\omega}$ is the angular velocity vector, the same for each $\underline{e_i}$ so that the axes rotate rigidly (We can check that $(\underline{e_i} \cdot \underline{e_j})^{\cdot} = 0$)

For an arbitrary vector $\underline{b}(t)$,

rate of change relative to non rotating axes →

rate of change relative to the rotating axes ↓ compensates for the motion of the axes ↓

$$\frac{d}{dt}\underline{b}(t) \equiv (b_i \underline{e_i})^{\cdot} = \dot{b}_i \underline{e_i} + b_i \underline{\omega} \times \underline{e_i} = \dot{\underline{b}}_{rot} + \underline{\omega} \times \underline{b} \quad (4.5$$

4.3 N2 in rotating axes

First calculate the relationship between the acceleration relative to rotating and non-rotating axes. We use (4.5) twice.

First set $\underline{r} = r_i \underline{e_i}$, so that

$$\underline{v} = \frac{d\underline{r}}{dt} = \underline{v}_{rot} + \underline{\omega} \times \underline{r} \qquad (\underline{v}_{rot} = \dot{r}_i \underline{e_i})$$

non rotating frame →

$$\text{Then } \underline{a} \equiv \frac{d\underline{v}}{dt} = (\ddot{r}_i \underline{e_i} + \underline{\omega} \times \underline{v}_{rot}) + \dot{\underline{\omega}} \times \underline{r} + \underline{\omega} \times (\underline{v}_{rot} + \underline{\omega} \times \underline{r}$$

$$\underline{a} = \underline{a}_{rot} + 2\underline{\omega} \times \underline{v}_{rot} + \underline{\omega} \times (\underline{\omega} \times \underline{r}) + \dot{\underline{\omega}} \times \underline{r}$$

Example Polar coordinates

$\underline{e_1} = \hat{\underline{r}}, \ \underline{e_2} = \hat{\underline{\theta}}, \ \underline{e_3} = \hat{\underline{z}}$

Then $\underline{r} = r\hat{\underline{r}}, \ \underline{\omega} = \dot{\theta}\hat{\underline{z}}$

$\underline{v}_{rot} = \dot{r}\hat{\underline{r}}, \ \underline{a}_{rot} = \ddot{r}\hat{\underline{r}}$

$\dot{\underline{\omega}} = \ddot{\theta}\hat{\underline{z}} \qquad \Rightarrow \underline{a} = \ddot{r}\hat{\underline{r}} + 2\dot{r}\dot{\theta}\hat{\underline{\theta}} - r\dot{\theta}^2\hat{\underline{r}} + r\ddot{\theta}\hat{\underline{\theta}}$

$\underline{a} = (\ddot{r} - r\dot{\theta}^2)\hat{\underline{r}} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\underline{\theta}}$ as expected (!)

(

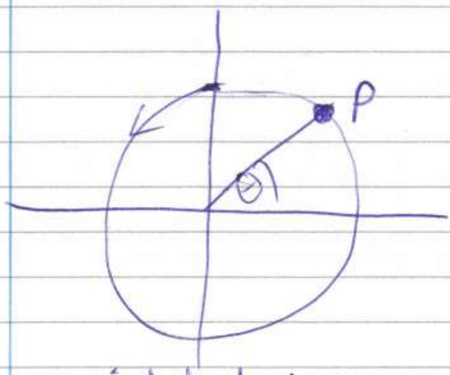
Assume the non-rotating axes are inertial, so that

$$\underline{F} = m\underline{a} \quad \text{and} \quad m\underline{a}_{rot} = \underline{F} - \underbrace{2m\,\underline{\omega}\times\underline{v}_{rot}}_{\text{Coriolis Force}} - \underbrace{m\,\underline{\omega}\times(\underline{\omega}\times\underline{r})}_{\text{Centrifugal Force outwards}} - m\,\dot{\underline{\omega}}\times\underline{r}$$

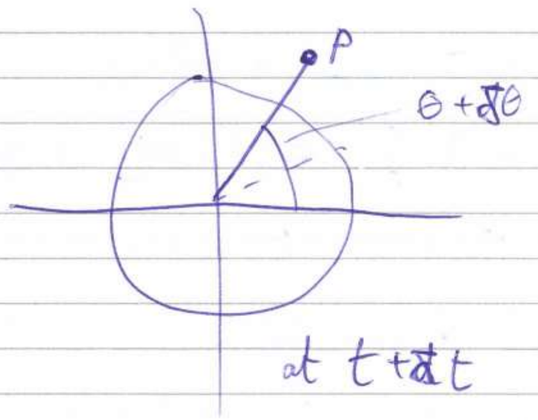
$\underline{F}$ is the actual physical force, the rest are called "fictitious forces".
The fictitious forces only appear when you try to apply N2 in the non-inertial frame. N2 does not apply but if you add the fictitious forces, the ~~moment~~ motion is as if N2 does apply.

4.5 Coriolis Force

This term affects motion in the plane orthogonal to $\underline{\omega}$. For motion radially outwards in the rotating frame, it compensates for the increase in the tangential velocity required to keep pace with the rotating frame.



inertial at t



at $t + \delta t$

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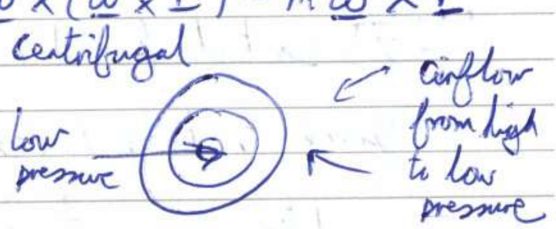
$$m\underline{a}_{rot} = \underline{F} - \underbrace{2m\,\underline{\omega}\times\underline{V}_{rot}}_{\text{coriolis}} - \underbrace{m\,\underline{\omega}\times(\underline{\omega}\times\underline{r})}_{\text{centrifugal}} - m\,\dot{\underline{\omega}}\times\underline{r}$$

4.5.1 Cyclones



Cyclone - An area of low pressure

Earth → 

$-2m\,\underline{\omega}\times\underline{V}_{rot}$ tends to deflect the air to its right.

So the airflow is anticlockwise. So airflow is ~~…~~ anticlockwise in the Northern hemisphere.

$e_i = \omega\times e_i$

4.5.2 Effect of the Coriolis Force on a falling particle - Handout 6

* 4.5.3 Foucault Pendulum

A long simple pendulum is suspended from a fixed pivot. Ignoring $\dot{\underline{\omega}}$ and $O(\omega^2)$ terms, the equation of motion is:

$$m\ddot{\underline{r}}_{rot} = -2m\,\underline{\omega}\times\dot{\underline{r}}_{rot} + m\underline{g} + \underline{T}$$

$\underline{r}$ is the position vector with respect to the centre of the Earth. $\underline{T}$ is the tension force.

Solving approximately shows that the plane of the pendulum swing rotates with period $\frac{2\pi}{\omega\sin\lambda}$, where λ is the latitude ~~…~~ which gives roughly 32 hours, in Paris.

(Gauss Bonnet says that the sum of all small angles $\propto$ to the area of the cap)

4.6 Centrifugal Force $-m\underline{\omega}\times(\underline{\omega}\times\underline{r})$

A particle fixed in the rotating frame must experience a physical force to keep it fixed. But $\underline{a}_{rot}=0$ and we add a fictitious force to balance the physical force to make it appear that N2 holds in the non inertial frame.

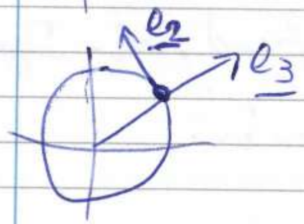
4.6.1 Apparent Gravity

Define 'apparent gravity' $\underline{g}'$ by $\underline{g}'=\underline{g}-\underline{\omega}\times(\underline{\omega}\times\underline{r})$

For a point at latitude λ we choose axes with

$\underline{e_3}$ - radially outwards, $\underline{e_2}$ - North, $\underline{e_1}$ - East

so that $\underline{g}=(0,0,-g)$



$\underline{r}=(0,0,R)$

$\underline{\omega}=(0,\cos\lambda,\sin\lambda)\,\omega$

and $\underline{\omega}\times(\underline{\omega}\times\underline{r})=R\omega^2\cos\lambda\,(0,\sin\lambda,-\cos\lambda)$

Thus $\underline{g}'$ makes an angle of

$\arctan\left(\frac{\omega^2R\cos\lambda\sin\lambda}{g-\omega^2R\cos^2\lambda}\right)$ with $\underline{g}$.

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Hand-out 6: Coriolis effect on a falling body

We consider the effect of the coriolis force on a particle dropped from a fixed point in the rotating frame of the Earth — the top of a tower, say (as in Galileo's experiment). In the rotating frame, the appropriate equation of motion, omitting the centrifugal and varying angular velocity terms (both of which are negligible unless the tower is enormous).

$$\ddot{\mathbf{r}} = \mathbf{g} - 2\boldsymbol{\omega} \times \dot{\mathbf{r}} \qquad (*)$$

where $\mathbf{r}$ and its derivatives are all relative to the rotating frame. We can integrate $(*)$ directly once:

$$\dot{\mathbf{r}} - \dot{\mathbf{r}}(0) = \mathbf{g}t - 2\boldsymbol{\omega} \times \mathbf{r} + 2\boldsymbol{\omega} \times \mathbf{r}(0). \qquad (**)$$

We are considering a dropped particle, so we take $\dot{\mathbf{r}}(0) = 0$. Let $\mathbf{r}(0) = \mathbf{r}_0$.

We could at this point simply lurch into components and integrate the system of first order equations but since we are already ignoring terms of $O(\omega^2)$ by omitting the centrifugal acceleration, we can do better. Substituting $(**)$ into $(*)$ gives

$$\ddot{\mathbf{r}} = \mathbf{g} - 2\boldsymbol{\omega} \times (\mathbf{g}t - 2\boldsymbol{\omega} \times (\mathbf{r} - \mathbf{r_0}))$$

and ignoring the last term $-2\boldsymbol{\omega} \times (\mathbf{r} - \mathbf{r_0})$, (which is small compared with $\mathbf{g}t$), we obtain

$$\ddot{\mathbf{r}} = \mathbf{g} - 2\boldsymbol{\omega} \times \mathbf{g}t.$$

This equation can be integrated twice directly:

$$\mathbf{r} = \tfrac{1}{2}\mathbf{g}t^2 - \tfrac{1}{3}\boldsymbol{\omega} \times \mathbf{g}t^3 + \mathbf{r}_0.$$

Now at last we take choose axes. We will assume for simplicity that our tower is at the equator. Our axial directions at the top or bottom of the tower are, as usual:

$$\mathbf{e}_1 \text{ easterly}; \quad \mathbf{e}_2 \text{ northerly}; \quad \mathbf{e}_3 \text{ radially outwards};$$

which form a right-handed set.

With respect to these axes,

$$\mathbf{g} = (0,0,-g), \quad \boldsymbol{\omega} = (0,\omega,0), \quad \mathbf{r}_0 = (0,0,R+h), \quad \mathbf{r} = (x,y,z)$$

where R is the radius of the Earth and h is the height of the tower (above the surface of the Earth). Thus

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \tfrac{1}{2}gt^2 \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} + \tfrac{1}{3}\omega g t^3 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ R+h \end{pmatrix} \qquad (\dagger)$$

To this approximation (ignoring the curvature of the Earth) the surface of the Earth is $z = R$.[1] Substituting $z = R$ into the third component of the $(\dagger)$ reveals that the approximate descent time is $\sqrt{2h/g}$, as in the non-rotating case. At this time,

$$x = \frac{2\sqrt{2}}{3}\frac{\omega h^{3/2}}{g^{1/2}}$$

which is the distance eastwards from the bottom of the tower at which the particle lands.[2]

This can easily be understood in the inertial (non-rotating) frame. Just before being dropped, the particle is at radius $(R+h)$ and co-rotating, so it has angular momentum per unit mass $(R+h)^2\omega$. As it falls, its angular momentum is conserved (the only force is central), so its speed v on landing is given by $vR = (R+h)^2\omega$. Therefore, its speed in the (eastward) direction of rotation increases from $(R+h)\omega$ to $(R+h)^2\omega/R$ and it gets ahead of the tower.

[1] The z axis is in fact tangent to the surface so for a very high tower we would have to take into account the curvature of the surface of the Earth to find the value of z that corresponded to hitting the ground again.

[2] About 55cm for a particle dropped from the Burj Dubai.

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Chapter 5 - Systems of Particles

5.1 Equations of Motion

We consider a system of n particles subject to external forces and internal forces from the other particles. The i^{th} particle has mass m_i and position vector (with respect to some arbitrary origin) $\underline{r}_i$ and equation of motion $m_i \ddot{\underline{r}}_i \equiv \dot{\underline{p}}_i = \underline{F}_i^e + \sum_{j \neq i} \underline{F}_{ij}$
where $\underline{F}_i^e$ is the external force on the i^{th} particle, and $\underline{F}_{ij}$ is the internal force on the i^{th} particle due to the j^{th}. This all takes place in an inertial frame

5.1.1 Total Momentum

We define the total momentum of the system, $\underline{P}$, by
$\underline{P} = \sum m_i \dot{\underline{r}}_i$ so that $\dot{\underline{P}} = \sum m_i \ddot{\underline{r}}_i = \sum_i \underline{F}_i^e + \sum_i \sum_{j=1}^{n} \underline{F}_{ij}$

$= \underbrace{\underline{F}^e}_{\text{total external force}} + \underline{0} \leftarrow$ by N3

Thus, if $\underline{F}^e = \underline{0}$, total momentum is conserved. If the external force is a uniform gravitational force, $\underline{F}_i^e = m_i \underline{g}$, then $\dot{\underline{P}} = \sum m_i \underline{g} = M\underline{g}$ where M is the total mass.

The centre of mass $\underline{R}$ is defined by $M\underline{R} = \sum m_i \underline{r}_i$

We have $M\ddot{\underline{R}} \equiv \dot{\underline{P}} = \underline{F}^e$

So the centre of mass moves as a single particle in external force field $\underline{F}^e$

5.1.2 Total Angular Momentum

We define the total angular momentum of the system $\underline{H}$ about an arbitrary origin $\underline{r} = \underline{0}$ by $\underline{H} = \sum \underline{r}_i \times \underline{p}_i$

using N2

So that $\underline{\dot{H}} = \underbrace{\sum \underline{\dot{r}}_i \times \underline{p}_i}_{\underline{0}} + \sum \underline{r}_i \times \underline{\dot{p}}_i = \sum \underline{r}_i \times \underline{F}_i^e + \sum\sum \underline{r}_i \times \underline{F}_{ij}$

$\underline{\dot{H}} = \underline{G}$ (the total external torque) $+ \sum\sum \underline{r}_i \times \underline{F}_{ij}$

For central forces, $\underline{F}_{ij} = f_{ij}(|\underline{r}_i - \underline{r}_j|)(\underline{r}_i - \underline{r}_j)$ where, by N3 $f_{ij} = f_{ji}$. In this case $\sum\sum \underline{r}_i \times \underline{F}_{ij} = \sum\sum -\underline{r}_i \times \underline{r}_j f_{ij} = 0$ and $\underline{\dot{H}} = \underline{G}$, the total external torque. In addition, if the external field is a uniform gravitational field, $\underline{F}_i^e = m_i \underline{g}$ and $\underline{G} = \sum \underline{r}_i \times (m_i \underline{g})$, $\underline{G} = M\underline{R} \times \underline{g}$. This means that the total external torque acts at the centre of gravity.

5.1.4 Centre of mass frame

It is often helpful to work in the frame with origin at $\underline{R}$. We define $\underline{y}_i = \underline{r}_i - \underline{R}$ so $\underline{y}_i$ is the position vector of the i^{th} particle with respect to $\underline{R}$. Then the $\underline{y}_i$'s satisfy $\sum m_i \underline{y}_i = \underline{0}$ and similarly $\sum m_i \underline{\dot{y}}_i = \underline{0}$ so the total momentum in the centre of mass frame, is $\underline{0}$. NB! The centre of mass frame is in general ($\underline{F}^e \neq \underline{0}$) non-inertial, so N2 does not apply.

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$\underline{H} = \sum m_i \underline{r}_i \times \dot{\underline{r}}_i = \sum m_i (\underline{y}_i + \underline{R}) \times (\dot{\underline{y}}_i + \dot{\underline{R}})$ In any Frame

$\underline{H} = \sum m_i \underline{y}_i \times \dot{\underline{y}}_i + \underbrace{\sum m_i \underline{y}_i}_{=0} \times \dot{\underline{R}} + \sum m_i \underline{R} \times \dot{\underline{y}}_i = 0$
$+ \sum m_i \underline{R} \times \dot{\underline{R}}$

$\underline{H} = \underline{H}_M + M \underline{R} \times \dot{\underline{R}}$

Total angular momentum about the centre of mass — Angular momentum of the centre of mass about the origin

Similarly, the total Kinetic Energy is defined by

$\frac{1}{2} \sum m_i \dot{\underline{r}}_i \cdot \dot{\underline{r}}_i$ can be expressed in the form

$\frac{1}{2} \sum m_i \dot{\underline{y}}_i \cdot \dot{\underline{y}}_i + \frac{1}{2} M \dot{\underline{R}} \cdot \dot{\underline{R}}$ ← Kinetic energy of the centre of mass

Total Kinetic Energy in the centre of mass frame

Surprisingly, $\underline{H}_M$ satisfies $\dot{\underline{H}}_M = \underline{G}_M$ ← Total Torque about the centre of mass

(just differentiate $\sum m_i \underline{y}_i \times \dot{\underline{y}}_i$)

Summary of Key Results

$\dot{\underline{P}} \equiv M\ddot{\underline{R}} = \underline{F}^e$ (inertial frame)

$= M\underline{g}$ (inertial frame and uniform gravity)

$\dot{\underline{H}} = \underline{G}$ (inertial frame, or Centre of Mass frame with central internal forces)

$= M\underline{R} \times \underline{g}$ (as above, with uniform gravity)

$\underline{P}_M = \underline{0}$ (Centre of Mass frame)

$\underline{H} = \underline{H}_M + M\underline{R} \times \dot{\underline{R}}$

$KE = KE_M + \frac{1}{2} M \dot{\underline{R}} \cdot \dot{\underline{R}}$

5.2 Two Body Problem

5.2.1 Equations of Motion

Let $\underline{R} = \frac{m_1\underline{r}_1 + m_2\underline{r}_2}{M}$, $\underline{r} = \underline{r}_1 - \underline{r}_2$

Assume $\underline{F}_i^e = \underline{0}$, so that $\ddot{\underline{R}} = \underline{0}$ and

$\ddot{\underline{r}} = \ddot{\underline{r}}_1 - \ddot{\underline{r}}_2 = \frac{\underline{F}_{12}}{m_1} - \frac{\underline{F}_{21}}{m_2}$

i.e. $\mu\ddot{\underline{r}} = \underline{F}$ $(\underline{F}_{12} = -\underline{F}_{21})$, μ, the reduced mass $= \frac{m_1 m_2}{m_1+m_2}$

If $\underline{F}$ is central, $\exists\ \phi(r)$ such that $\underline{F} = -\underline{\nabla}\phi$

We define the total energy E by $E = \frac{1}{2}\sum m_i \dot{\underline{r}}_i\cdot\dot{\underline{r}}_i + \phi(r)$

$E = \frac{1}{2}\sum m_i \dot{\underline{y}}_i\cdot\dot{\underline{y}}_i + \frac{1}{2}M\dot{\underline{R}}\cdot\dot{\underline{R}} + \phi(r)$

$E = \frac{1}{2}\mu\dot{\underline{r}}\cdot\dot{\underline{r}} + \frac{1}{2}M\dot{\underline{R}}\cdot\dot{\underline{R}} + \phi(r)$ (using $\underline{y}_1 = \frac{m_2}{M}\underline{r}$, etc)

E is conserved: $\frac{dE}{dt} = \mu\dot{\underline{r}}\cdot\ddot{\underline{r}} + M\dot{\underline{R}}\cdot\underbrace{\ddot{\underline{R}}}_{0} + \frac{d\phi}{dt}$

$\frac{dE}{dt} = \dot{\underline{r}}\cdot\underline{F} + \nabla\phi\cdot\dot{\underline{r}} = \dot{\underline{r}}\cdot\underline{F} - \underline{F}\cdot\dot{\underline{r}} = \underline{0}$ (as required)

For gravitating bodies, like the Sun and Jupiter, we have

$\mu\ddot{\underline{r}} = -\frac{GM_1M_2}{r^3}\underline{r}$

Since $\underline{y}_1 \propto \underline{r}$,

$\ddot{\underline{y}}_1 = k\frac{\underline{y}_1}{|\underline{y}_1|^3}$ just as in the case of a single body moving in the Gravitational Field of a fixed body. Thus, for two particles move on conic sections with the centre of mass as focus.

Hand-out 7: Drum majorette's baton

We model the baton as a light rod of length ℓ with masses m_1 and m_2 attached to the ends. What happens when the baton is thrown up into the air?

Let $\mathbf{y}_1$ and $\mathbf{y}_2$ be the position vectors of the two masses with respect to the centre of mass. Then

$$m_1\mathbf{y}_1 + m_2\mathbf{y}_2 = 0.$$

Setting $|\mathbf{y}_i| = y_i$, we have $y_1 + y_2 = \ell$ and (from the above equation)

$$m_1 y_1 = m_2 y_2.$$

The external force on the system is the uniform gravitational field $\mathbf{g}$. The internal force between the particles is the stress or tension in the light rod. This force is central: it acts in the direction of the vector joining the two particles.

Let $\mathbf{R}$ be the position of the centre of mass. We know that

$$M\ddot{\mathbf{R}} = \mathbf{F}^e = m_1\mathbf{g} + m_2\mathbf{g} = M\mathbf{g}$$

so the centre of mass moves exactly as if it were a single particle of mass M in a gravitational field.

Since the rod is rigid, the two masses are rotating about the centre of mass with the same angular velocity $\boldsymbol{\omega}$. The velocity of the mass m_i with respect to the centre of mass is therefore $\boldsymbol{\omega} \times \mathbf{y}_i$ and

$$\mathbf{H}_M = m_1\mathbf{y}_1 \times (\boldsymbol{\omega} \times \mathbf{y}_1) + m_2\mathbf{y}_2 \times (\boldsymbol{\omega} \times \mathbf{y}_2)$$

The axis of rotation is perpendicular to the rod; since the rod is thin and the masses are particles they cannot rotate about an axis parallel to the rod. Expanding the vector products in the above equation and using $\boldsymbol{\omega} \cdot \mathbf{y}_i = 0$ shows that

$$\mathbf{H}_M = (m_1 y_1^2 + m_2 y_2^2)\boldsymbol{\omega}. \qquad (*)$$

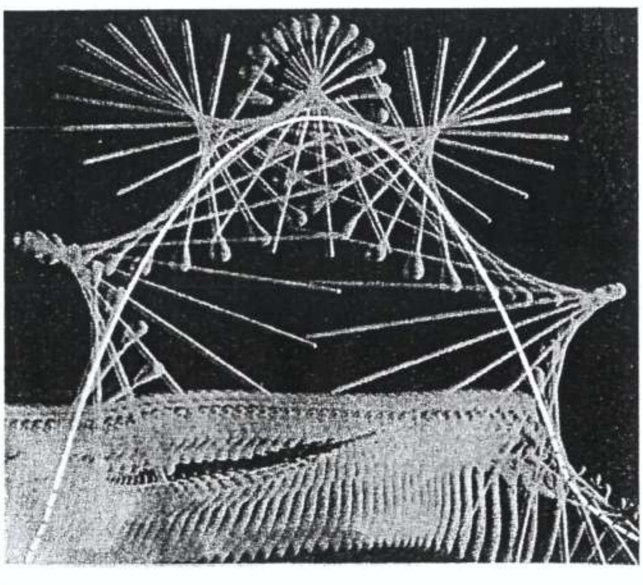
The centre of mass is fixed in the rod, so y_1^2 and y_2^2 are constant.

The gravitational torque $\mathbf{G}_M$ about the centre of mass is

$$\mathbf{y}_1 \times (m_1\mathbf{g}) + \mathbf{y}_2 \times (m_2\mathbf{g}) = (m_1\mathbf{y}_1 + m_2\mathbf{y}_2) \times \mathbf{g} = \mathbf{0}.$$

Thus the angular momentum about the centre of mass of the baton is constant and, from $(*)$, $\boldsymbol{\omega}$ is constant. Hence $\dot{\theta}$ is constant in the motion, where θ is the angle the baton makes with the vertical, and $|\dot{\theta}| = |\boldsymbol{\omega}|$.

The time lapse photograph below shows this nicely: the centre of mass moves on a parabola and the angle of the rod changes by the same amount between each exposure.



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5.4 Rockets

5.4.1 Rocket Equation

We consider a rocket moving in one dimensions. At time t, its total mass (including the fuel and body) $m(t)$ and its velocity is $v(t)$. It ejects exhaust gases at speed $-u$ relative to the rocket. We work in the inertial frame in which the rocket is instantaneously at rest at a given time t, and use $\frac{d\underline{p}}{dt} = \underline{F}^e$. At time t we have $p(t) = 0$. At time $t+\delta t$, $p(t+\delta t) = (-\delta m)(-u) + (m+\delta m)\delta v$

$(-\delta m)$ is the mass of exhaust gas ejected in δt. Then

$$\frac{dp}{dt} = \boxed{u\frac{dm}{dt} + m\frac{dv}{dt} = F^e} \quad \text{The Rocket Equation}$$

If $F^e = 0$ and u is constant we can integrate:

$$u\int \frac{dm}{m} = -\int dv \Rightarrow v(t) = v(0) + u\log\frac{m(0)}{m(t)}$$

(Tsiolkovski Equation, 1903)

5.4.2 Rocket with Linear Drag

Assume $\frac{dm}{dt} = -\alpha$ (constant) and u is constant

$$u(-\alpha) + m\frac{dv}{dt} = -kv$$

We could set $m(t) = m_0 - \alpha t$ and integrate to obtain $v(t)$, or we could use the chain rule to obtain $v(m)$:

$$-\alpha u + \frac{dv}{dm}(m \times -\alpha) = -kv \Rightarrow \frac{\alpha u}{k}\left(1 - \left(\frac{m}{m_0}\right)^{k/\alpha}\right) = v(m)$$

$v(m_0) = 0$

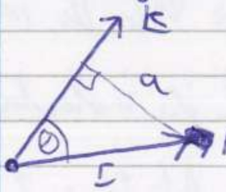
The max speed, if the rocket is all propellant, is $\frac{\alpha u}{k}$

5·5 Moments of Inertia

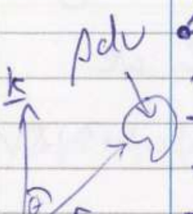
5·5·1 Definition We consider a particle of mass m rotating about an axis $\underline{k}$ with angular velocity $\underline{\omega}$ $(\underline{\omega} = \underline{k}\,\omega)$

$$KE = \tfrac{1}{2} m \underline{v}\cdot\underline{v} = \tfrac{1}{2} m (\underline{\omega}\times\underline{r})\cdot(\underline{\omega}\times\underline{r}) = \tfrac{1}{2} m \omega^2 (\underline{k}\times\underline{r})\cdot(\underline{k}\times\underline{r})$$

We define the moment of inertia by $I = m(\underline{k}\times\underline{r})\cdot(\underline{k}\times\underline{r})$



$I = ma^2$. For a system of particles, all rotating about $\underline{k}$ with angular velocity $\underline{\omega}$, $I = \sum m_i (\underline{k}\times\underline{r}_i)\cdot(\underline{k}\times\underline{r}_i)$



$I = \sum m_i a_i^2$. For a solid body, $\sum \to \int$, so

$$I = \int_V (r\sin\theta)^2 \rho\, dV$$

5·5·2 Examples

i) Thin uniform rod, axis perpendicular to the rod, through one end

$$I = \int_0^L x^2 \rho\, dx = \tfrac{1}{3} L^3 \rho = \tfrac{1}{3} M L^2$$

iv) Uniform Disc, axis ⊥ to the plane of the disc, through a point on the circumference

$$I = \int_A (r^2 + a^2 - 2ar\cos\theta)\rho\, dA = \int_0^{2\pi}\int_0^a (r^2 + a^2 - 2ar\cos\theta)\rho\, r\, dr\, d\theta$$

$$I = \tfrac{3}{2} M a^2$$

vi) Uniform sphere, on an axis through the centre

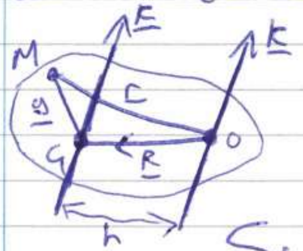
$$I = \int_0^{2\pi}\int_0^{\pi}\int_0^a r^2 \sin^2\theta\, \rho\, r^2 \sin\theta\, dr\, d\theta\, d\varphi = \tfrac{2}{5} M a^2$$

5.5.3 Parallel Axis Theorem

For the i^{th} particle $I = m(\underline{k}\times\underline{r})\cdot(\underline{k}\times\underline{r})$ $(m_i = m, \underline{r}_i = \underline{r}$

$I = m[\underline{k}\times(\underline{R}+\underline{y})]\cdot[\underline{k}\times(\underline{R}+\underline{y})]$ with $\underline{R}$ the centre of mass

$I = m(\underline{k}\times\underline{R})\cdot(\underline{k}\times\underline{R}) + m(\underline{k}\times\underline{y})\cdot(\underline{k}\times\underline{y}) + 2m(\underline{k}\times\underline{y})\cdot(\underline{k}\times\underline{R})$



$I = mh^2 + \underset{\nearrow}{I'} + [\text{linear } y \text{ terms}]$

moment of inertia about axis through G.

Summing over i, we find, for the system

$I = Mh^2 + I'$ since $\sum m_i \underline{y}_i = \underline{0}$

5.7 Rigid bodies

5.7.1 Velocity

For a rigid body rotating about a fixed axis, each point in the body rotates with the same angular velocity. For general motion, the velocity of the i^{th} particle can be expressed in the form

$\dot{\underline{r}}_i = \underset{\nearrow}{\dot{\underline{Q}}} + \underline{\omega}\times(\underline{r}_i - \underline{Q})$

velocity of a point fixed in the body:

The angular velocity $\underline{\omega}$ is independent of the choice of $\underline{Q}$:

(i) $\dot{\underline{r}}_i = \dot{\underline{Q}}_1 + \underline{\omega}_1\times(\underline{r}_i - \underline{Q}_1)$

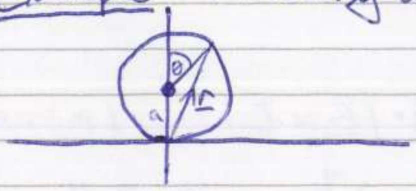
(ii) $\dot{\underline{r}}_i = \dot{\underline{Q}}_2 + \underline{\omega}_2\times(\underline{r}_i - \underline{Q}_2)$

(iii) $\dot{\underline{Q}}_1 = \dot{\underline{Q}}_2 + \underline{\omega}_2\times(\underline{Q}_1 - \underline{Q}_2)$

(i) − (ii) + (iii) $\Rightarrow \underline{\omega}_1\times(\underline{r}_i - \underline{Q}_1) = \underline{\omega}_2\times(\underline{r}_i - \underline{Q}_1)$

This is true for any $\underline{r}_i$, so $\underline{\omega}_1 = \underline{\omega}_2$

Example A rolling disc. The angular velocity is $\dot{\theta}\underline{k}$



with $\underline{k}$ into the paper. The velocity of a point P on the circumference is $\dot{\theta}\,\underline{k}\times\underline{r}$ where $\underline{r}=\underline{0}$ is the instantaneous point of contact

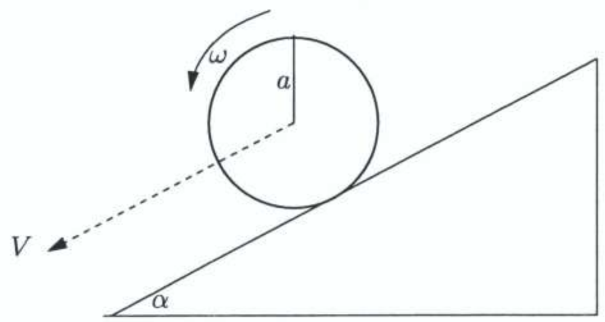
Mathematical Tripos Part IA
Dynamics and Relativity

Lent term 2010
Dr S.T.C. Siklos

Hand-out 8: Rolling disc

A uniform disc of mass m and radius a rolls without slipping down a line of greatest slope of an inclined plane of angle α. The plane of the disc is vertical. The moment of inertial of the disc about and axis through its centre perpendicular to the plane of the disc is I.

The motion of the disc consists of the linear motion of the centre of mass, which moves with speed V down the plane, and rotation about the centre of mass with angular speed ω, as shown. The angular velocity vector sticks out of the paper (right-handed corkscrew rule).



The point on the circumference of the disc that is instantaneously in contact with the plane is instantaneously at rest, because of the no-slip condition. This means that V and ω are related by

$$V - a\omega = 0\,.$$

This comes from $\mathbf{V} + \boldsymbol{\omega} \times \mathbf{y} = \mathbf{0}$, were $\mathbf{y}$ is the position vector of the instantaneous point of contact with respect to the centre of the disc. Taking instead the instantaneous point of contact as the origin, this equation says that the velocity the centre of mass is due to the rotation with angular velocity of $\boldsymbol{\omega}$ about the point of contact.

Using conservation of energy

The kinetic energy (using the result that the total KE is 'KE of centre of mass' plus KE relative to centre of mass) of the disc is

$$\tfrac{1}{2}mV^2 + \tfrac{1}{2}I\omega^2 = \tfrac{1}{2}mV^2 + \tfrac{1}{2}\,I(V/a)^2 = \tfrac{1}{2}(I/a^2 + m)V^2.$$

Let x be the distance down the plane that the disc has rolled at time t, so that $\dot{x} = V$. Then conserving energy gives

$$\tfrac{1}{2}(I/a^2 + m)\dot{x}^2 - mgx\sin\alpha = \text{constant}$$

(The minus sign arises because x measure distance *down* the plane.) Curiously, the quickest way to integrate this is to differentiate it and cancel a factor of $\dot{x}$, leaving a linear equation:

$$(I/a^2 + m)\ddot{x} = mg\sin\alpha$$

which can then be integrated twice. We see that the acceleration of a rolling disc is less, by a factor of $1 + I/ma^2$, than that of the same disc sliding without rolling down the same plane.

Using forces

The external forces on the disc are shown in the diagram below.

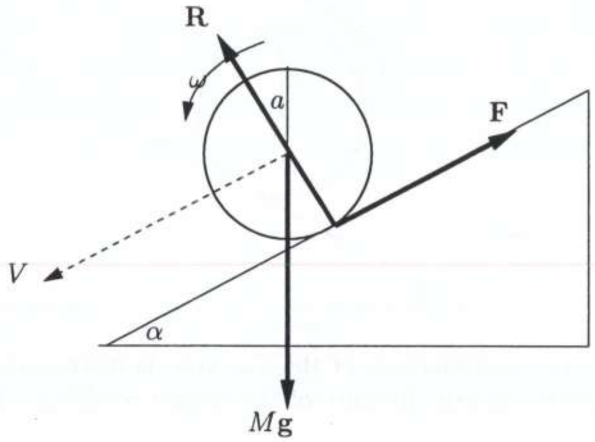
Again regarding the disc as a system of particles, we have the general results

$$M\ddot{\mathbf{R}} = \mathbf{F}^e$$

where M is the total mass, $\mathbf{R}$ is the position of the centre of mass and $\mathbf{F}^e$ is the sum of the external forces, and

$$\frac{d\mathbf{H}_M}{dt} = \mathbf{G}_M$$

where $\mathbf{H}_M$ is the total angular momentum about the centre of mass and $\mathbf{G}_M$ is the total external torque about the centre of mass (i.e. the total moment of the external forces).



The forces acting are gravity, friction and normal reaction. Thus

$$m\dot{V} = Mg\sin\alpha - F$$
$$I\dot{\omega} = aF.$$

Eliminating F from these equations, and using $\omega = V/a$ gives

$$(m + I/a^2)\dot{V} = mg\sin\alpha \qquad (\dagger)$$

which is the same equation as motion as that derived using conservation of energy.

We could have obtained this same result more directly using again

$$\dot{\mathbf{H}} = \mathbf{G}$$

where now the angular momentum and the torque are about the point of contact between the disc and the plane. Again $H = I'\omega$, but I' is the moment of inertial of the disc about an axis pointing out of the paper and passing through the point of contact, which by the parallel axis theorem is given by

$$I' = I + ma^2.$$

This gives the same equation as (†), since the shortest distance between the line of action of the force of gravity acting through the centre of the disc and the point of contact is $a\sin\alpha$.

Note that the ω in this calculation is the same as the ω that led to (†), because angular velocity is the same for all points on the disc.

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Dynamics and Relativity (18)

5·7·2 Angular Momentum

For a single particle rotating about an axis $\underline{k}$

$$\underline{H} = \underline{r} \times (m\dot{\underline{r}}) = m\underline{r} \times (\underline{r} \times \underline{\omega}) = m\omega\, \underline{r} \times (\underline{r} \times \underline{k})$$

Then $\underline{H} \cdot \underline{k} = m\omega\, \underline{r} \times (\underline{r} \times \underline{k}) \cdot \underline{k} = m\omega\, (\underline{r} \times \underline{k}) \cdot (\underline{r} \times \underline{k}) = \omega I$

Summing over all particles gives $\boxed{\underline{H} \cdot \underline{k} = I\omega}$ for a rigid body

If $\underline{k}$ is constant, $\frac{d}{dt}(\underline{H} \cdot \underline{k}) = \frac{d\underline{H}}{dt} \cdot \underline{k} = I\frac{d\omega}{dt} = \underline{G} \cdot \underline{k}$

using $\frac{d\underline{H}}{dt} = \underline{G}$

6 Special Relativity

6·1 Basic Concepts

6·1·1 Comparison with Newtonian Dynamics

Newtonian Dynamics	Special Relativity
1. There are a set of frames (Inertial Frames) in which Newton's 1st Law holds	Inertial Frames are the same in Special Relativity.
2i) Inertial Frames are related by Galilean Transformations. OR	Inertial Frames are related by Lorentz Transformations.
2ii) Time is absolute	The speed of light is the same in all inertial frames
3 Newton's Second Law holds in all inertial frames (Principle of Galilean Relativity)	Newton's Second Law holds in all inertial frames, after being suitably modified (Principle of Special Relativity)

We can extend the Principle of Special Relativity to include other laws of physics.

6.1.2 Consequences of Special Relativity

i) Simultaneity is frame dependent.

ii) Length contraction and Time dilation

iii) A New velocity addition law is necessary

iv) ~~A~~ Modified definitions for energy, momentum and other quantities are needed.

6.1.3 The need for Special Relativity

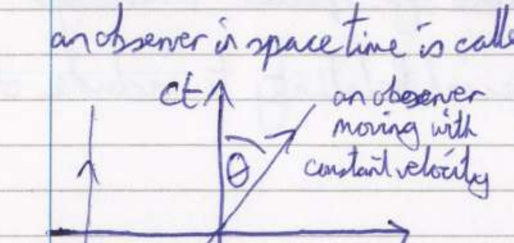
Galilean Relativity was found to be inadequate for both experimental and theoretical reasons, such as:

i) Measurements of the speed of light, c, by Michaelson and Morley in 1887 showed that it was frame independent, when Galilean Relativity gives $c' = c \pm v$ when $x' = x + vt$

ii) ~~[illegible]~~ Maxwell's equations, which govern the propagation of electromagnetic waves, are not invariant under Galilean Transformations.

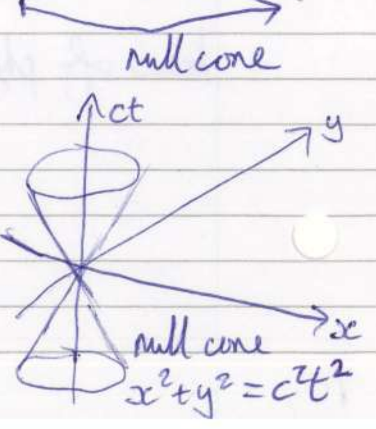
6.2 Spacetime Diagrams

Convention: Time is on the vertical axis, in units ct (length), so that rays of light are at 45° to the vertical. In 2D space with time, these form the null cone. The trajectory of an observer in spacetime is called a <u>world line</u>.



where $v < c$ so

$\theta < 45°$



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6.3 Lorentz Transformations

6.3.1 Definition

Let S be a frame with coordinates (ct, x), and let S' be a frame with coordinates (ct', x') moving with velocity v relative to S. The Lorentz Transformation relating S and S' is given by $x' = (x - vt)\gamma$ and $t' = (t - \frac{vx}{c^2})\gamma$

where $\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \geq 1$

Notes

i) As $\frac{v}{c} \to 0$, we obtain the Newtonian Limit, $\gamma \to 1$ and the Lorentz Transformation tends towards a Galilean Transformation.

ii) The Inverse Transformation is $x = (x' + vt')\gamma$, $t = (t' + \frac{vx'}{c^2})\gamma$ (by doing the algebra or observing that $x \leftrightarrow x'$, $t \leftrightarrow t'$, $v \to -v$)

iii) Lorentz Transformations are linear in x and t so they hold infinitesimally

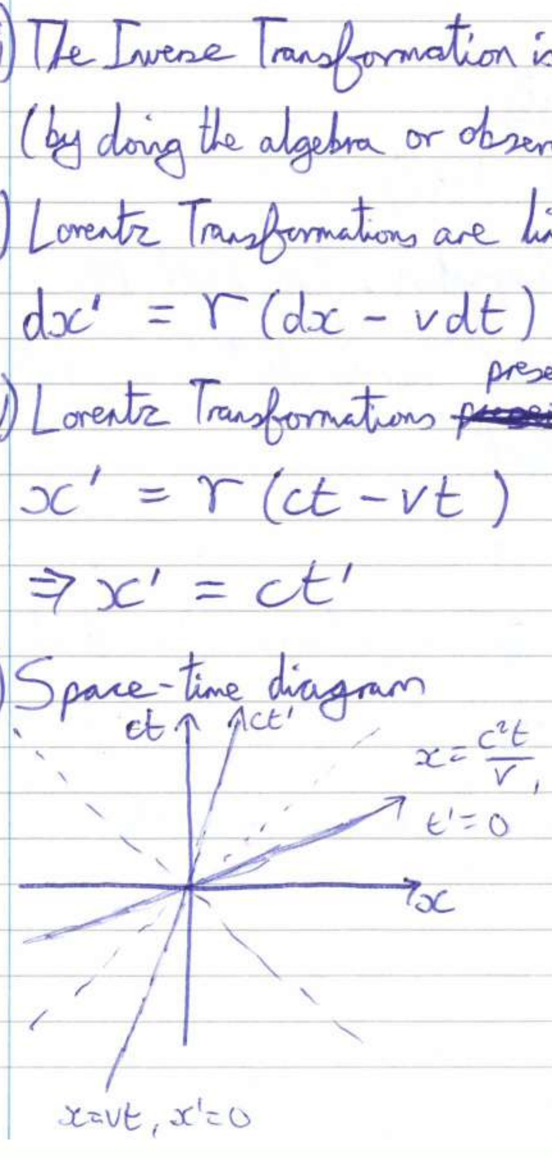
$dx' = \gamma(dx - v\,dt) \qquad dt' = \gamma(dt - \frac{v}{c^2}dx)$

iv) Lorentz Transformations ~~preserve~~ preserve the speed of light. If $x = ct$ in S,

$x' = \gamma(ct - vt) \qquad t' = \gamma(t - \frac{v}{c^2}ct)$

$\Rightarrow x' = ct'$

v) Space-time diagram



NB: The ct' axis makes the same angle with the ct as the x' axis makes with the x axis.

6.3.2 Matrix Representation (of Lorentz Transformations)

We have $\begin{pmatrix} ct' \\ x' \end{pmatrix} = \begin{pmatrix} \gamma & -\frac{\gamma v}{c} \\ -\frac{\gamma v}{c} & \gamma \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix} = \begin{pmatrix} \cosh\beta & -\sinh\beta \\ -\sinh\beta & \cosh\beta \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix}$

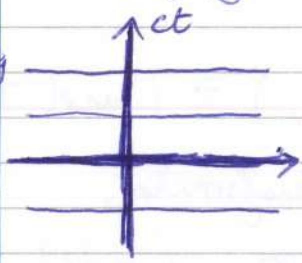
where $\tanh\beta = \frac{v}{c}$, a hyperbolic rotation. $= \underline{L}(\beta) \begin{pmatrix} ct \\ x \end{pmatrix}$

The Lorentz Transformations form a group with $L(\beta_2)L(\beta_1) = L(\beta_2+\beta_1)$

6.3.3 Simultaneity

To an observer in S, the line $t = t_1$ (say) joins all points that are simultaneous with t_1. To an observer in S', all points simultaneous with t_2 (say) all lie on the line $t' = t_2$, i.e. $\gamma(t - \frac{vx}{c^2}) = t_2$

Simultaneity in S



Simultaneity in S'

6.4 Time dilation and length contraction

6.4.1 Muon decay (Rossi and Hall, 1941)

Muons are created in the upper atmosphere. The half life of muons is very short. Muons "should" nearly all decay before reaching ground level, but they don't.

The explanation for this:

i) In the lab frame, the moving muon's time runs slower (time dilation) so the half life is longer than we expect.

ii) In the muon frame, the atmosphere's height is contracted (length contracti

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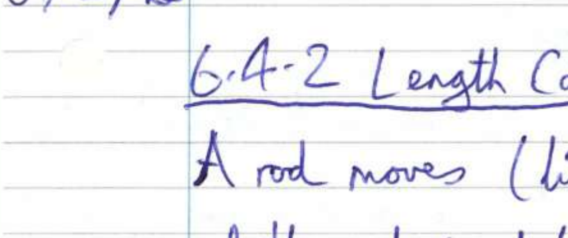
6.4.2 Length Contraction

A rod moves (like a javelin) at velocity v in S. The length of the rod is L' measured in the rest frame S' of the rod, and its length is L in S.



In S, the world line of the front of the rod is $x = vt + L$, and in S' it is $x' = L'$ $\xrightarrow[\text{Transformation}]{\text{Lorentz}}$ $\gamma(x - vt) = L'$

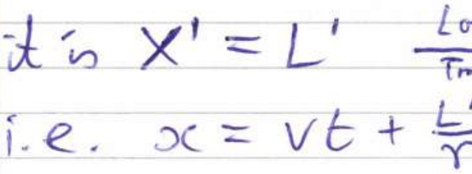
i.e. $x = vt + \frac{L'}{\gamma}$ $\Rightarrow L = L'/\gamma \leq L'$ (length contraction)

So the length of a moving rod is less than its length in its rest frame.

$x' = \gamma(x - vt)$

6.4.3 Time Dilation

A clock moves at velocity v in S. The time between ticks in the rest frame S' is $\Delta t'$, and in the frame S, is Δt.



Choose axes such that the origin of both S and S' are at a given tick. The second tick is at $(c\Delta t', 0)$ in S' and at $(c\Delta t, v\Delta t)$ in S.

But $(c\Delta t', 0)$ $\xrightarrow[\text{Transformation}]{\text{Lorentz}}$ $(\gamma c\Delta t', \gamma v \Delta t')$

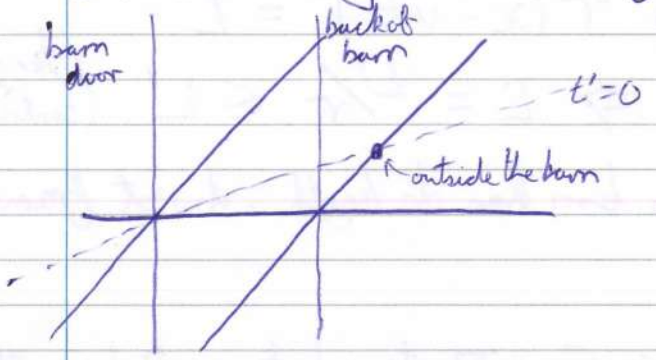
$\Rightarrow \Delta t = \gamma \Delta t' \geq \Delta t'$ (time dilation)

So moving clocks run more slowly.

$t' = \gamma\left(t - \frac{vx}{c^2}\right)$

6.4.4 The ladder-and-barn paradox

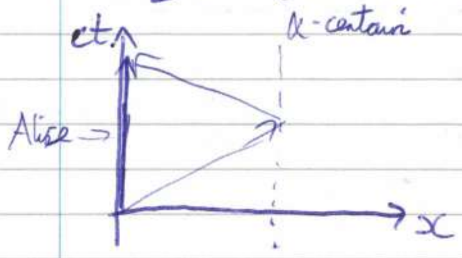
A builder carrying a ladder of length $2L$ runs with velocity $\frac{\sqrt{3}}{2}c$ (where $\gamma = 2$) towards a barn of length L. In the barn frame, the ladder is contracted to length L and fits in the barn. In the ladder frame, the barn is contracted to length $\frac{1}{2}L$, and the ladder does not fit. These statements are not contradictory because 'fit' depends on simultaneity and so is frame dependent.



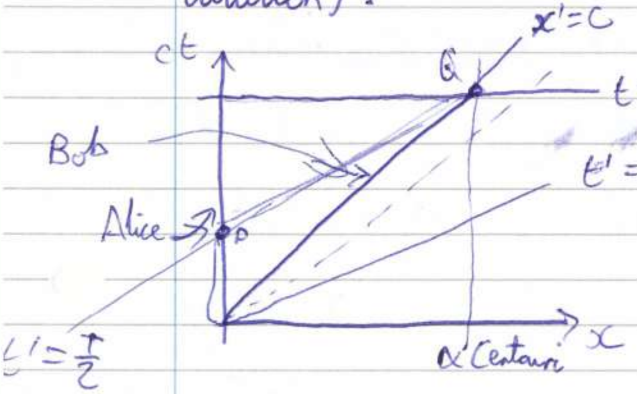
6.4.5 The twin paradox

6.4.5 The twins Paradox

Bob goes to α-Centauri, ~~while~~ and Alice stays at home. Bob's speed is $\frac{\sqrt{3}}{2}c$ (so that $\gamma = 2$). At their reunion, Alice has aged by $2T$ (say) and Bob by T (time dilation).



The situation is NOT symmetric as Bob has to accelerate at α-Centauri in order to return to Alice. But Alice's frame is inertial. Suppose Bob sets off at birth. For the outward journey, Alice will say "Bob arrived when I was T years old and his age was $\frac{T}{2}$" (time dilation). Bob will say "I arrived when I was $\frac{T}{2}$ and Alice's age was $\frac{T}{4}$" (time dilation).



In Bob's Frame, P is simultaneous with his arrival at α-Centauri.

How old is Alice at P?

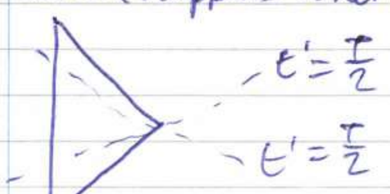
Exercise: i) Find the coordinates of Q in S.

ii) Lorentz Transform to find the coordinates of Q in S'.

iii) Write down the equation of the line PQ in S'.

iv) Transform to S and show that P is $(\frac{1}{4}cT, 0)$ in S.

What happens when Bob turns around?



According to Bob, Alice ages from $\frac{1}{4}T$ to $\frac{7}{4}T$ while he is turning around.

6·5 Velocity Transformation

S' moves at constant velocity v with respect to S. A particle moves with velocity u in S, and u' in S'. The world line of the particle in S is $x = ut$, and in S', $x' = u't'$.

$$u' = \frac{x'}{t'} = \frac{\cancel{\gamma}(x - vt)}{\cancel{\gamma}(t - \frac{v}{c^2}x)} = \frac{u - v}{1 - \frac{uv}{c^2}} \quad \text{Velocity Transformation law}$$

If $v = c$, $c' = \frac{u - c}{1 - \frac{u}{c}} = -c$

If $u = c$, $c' = \frac{c - v}{1 - \frac{v}{c}} = c$

Setting $\frac{u}{c} = \tanh\beta$, $\frac{u'}{c} = \tanh\beta'$ and $\frac{v}{c} = \tanh\alpha$, we see that $\beta' = \beta - \alpha$

6·6 Proper Time

Let (ct, x) and $[c(t+dt), x+dx]$ be the coordinates of events on the world line of an observer. We define the <u>proper time</u> interval between these points $d\tau$ by $c^2 d\tau^2 = c^2 dt^2 - dx^2$, and $d\tau > 0$ if $dt > 0$. The proper time interval is Lorentz invariant, the same in all inertial frames.

After a Lorentz Transformation

$$c^2 d\tau'^2 = c^2 dt'^2 - dx'^2 = c^2 dt^2 - dx^2 = c^2 d\tau^2$$ see 6·8.

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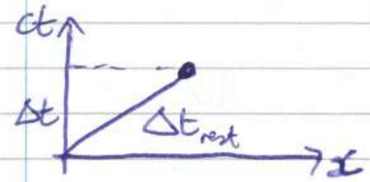
In the rest frame of the observer $dx = 0$,

and $d\tau = dt_{rest}$ $\qquad (c^2 d\tau^2 = c^2 dt^2 - dx^2$

so τ measures time in the observer's rest frame. Note that in general,

$d\tau = \sqrt{dt^2 - \frac{dx^2}{c^2}} \leq dt \quad \Rightarrow dt_{rest} \leq dt$ (time dilation)



$\Delta t \geq \Delta t_{rest}$

Using matrices: Minkowski metric

$$c^2 d\tau^2 = (cdt \;\; dx)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} cdt \\ dx \end{pmatrix}$$

Cartesian Form:

$$d\underline{x} \cdot d\underline{x} = (dx \;\; dy)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} dx \\ dy \end{pmatrix} \quad \text{Euclidean Metric}$$

$$d\underline{x} \cdot d\underline{x} = (dr, d\theta)\begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}\begin{pmatrix} dr \\ d\theta \end{pmatrix} \quad \text{in polar coordinates}$$

6.8 Four vectors

6.8.1 Definitions

In 1+3 dimensions, we write the position vector $\underline{X}$ as

$$\underline{X} = \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} ct \\ \underline{x} \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \text{ Minkowski Metric}$$

η We define a Lorentz Transformation by $L^T \eta L = \eta$

(L is the 4×4 Lorentz Transformation Matrix). We consider only Lorentz Transformations of the form
$$L = \begin{pmatrix} \gamma & -\frac{\gamma v}{c} & 0 & 0 \\ -\frac{\gamma v}{c} & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
(by choice of axes)

L defines a transformation from S to S':

$\underline{X}' = L\underline{X}$. A scalar, or Lorentz Invariant under Lorentz Transformati

1 is a quantity that is unchanged by Lorentz Transformations

A 4-vector is any quantity that transforms according to $V' = LV$.
Given any 4 vectors $\underline{V} = \begin{pmatrix} V_0 \\ \underline{v} \end{pmatrix}$ and $\underline{W} = \begin{pmatrix} W_0 \\ \underline{w} \end{pmatrix}$, we define the <u>scalar product</u>:

$$V \cdot W = V^T \eta W = V_0 W_0 - \underline{v} \cdot \underline{w}$$

$V \cdot W$ is a scalar: $(V \cdot W)' = V' \cdot W' = (LV)^T \eta (LW)$

$$(V \cdot W)' = V^T L^T \eta L W = V^T \eta W = V \cdot W$$

(using the definition of Lorentz Transformations)

In particular, $V \cdot V$ is invariant. We say that V is:
'timelike', 'spacelike', 'null' (light like)
$V \cdot V > 0$ $\quad$ $V \cdot V < 0$ $\quad$ $V \cdot V = 0$

We define infinitesimal proper time between events at X and $X + dX$ by $c^2 d\tau^2 = dX \cdot dX = c^2 dt^2 - d\underline{x} \cdot d\underline{x}$ $\quad X = \begin{pmatrix} ct \\ \underline{x} \end{pmatrix}$
assuming that dX is not spacelike (Take $d\tau > 0$ when $dt > 0$). $d\tau^2$ is Lorentz invariant by definition.

<u>6.8.2 4-Velocity</u>

We define the 4-Velocity of a particle with worldline $X(\tau)$ by $U = \frac{dX}{d\tau}$ which is a 4 vector because dX is a 4 vector and $d\tau$ is a scalar. We have

$$U = \frac{dX}{d\tau} = \frac{dX}{dt}\frac{dt}{d\tau} = \frac{dt}{d\tau}\begin{pmatrix} c \\ \underline{v} \end{pmatrix}$$

where $\underline{v}$ is the three-velocity of the particle.

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$U = \frac{dt}{d\tau}\begin{pmatrix} c \\ \underline{v} \end{pmatrix}$

In the rest frame of the particle, $\underline{v} = \underline{0}$ and $\frac{dt}{d\tau} = 1$, so

$U = \begin{pmatrix} c \\ \underline{0} \end{pmatrix}$

In the rest frame $U \cdot U = c^2$ and in the general frame,

$U \cdot U = \left(\frac{dt}{d\tau}\right)^2 (c^2 - v^2)$

$U \cdot U$ is invariant so $c^2 = \left(\frac{dt}{d\tau}\right)^2 (c^2 - v^2)$

$\Rightarrow \frac{dt}{d\tau} = \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}} = \gamma$

and $U = \begin{pmatrix} \gamma c \\ \gamma \underline{v} \end{pmatrix}$

6.8.3 Momentum 4-Vector

Let m be the rest mass of a particle moving with 4-velocity U. We define the 4-momentum P by:

$P = mU = \begin{pmatrix} m\gamma c \\ m\gamma \underline{v} \end{pmatrix} = \begin{pmatrix} E/c \\ \underline{p} \end{pmatrix}$

where $E = m\gamma c^2$ is the relativistic energy, and $\underline{p} = m\gamma \underline{v}$ is the relativistic 3-momentum.

In the rest frame, $P = (mc, \underline{0})$ and therefore $P \cdot P = m^2c^2$

In the general inertial frame, $P \cdot P = \frac{E^2}{c^2} - \underline{p} \cdot \underline{p}$

so $\boxed{E^2 = p^2c^2 + m^2c^4}$

which could have been obtained by eliminating γ from

$E = m\gamma c^2$, $p = m\gamma \underline{v}$ and $\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}}$

Note that in the Newtonian Limit, $\left|\frac{v}{c}\right| \ll 1$

$E = m\left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}} c^2 = mc^2 + \frac{1}{2}mv^2 + \ldots$

with mc^2 the rest energy, $\frac{1}{2}mv^2$ the Newtonian Kinetic Energy.

6·8·5 Massless Particles

h is Planck's Constant

For a photon, $E = h\nu$ $\quad$ ν is the frequency

$p = \frac{h\nu}{c}$, from wave-mechanics.

Setting $P = \begin{pmatrix} E/c \\ \underline{p} \end{pmatrix} = \frac{h\nu}{c}\begin{pmatrix} 1 \\ \underline{k} \end{pmatrix}$ where $\underline{k}$ is a unit 3-vector, the wave vector

which can be shown (from wave mechanics) to transform as a 4-vector under Lorentz Transformations. In general,

$P \cdot P = m^2c^2$; for photons, $P \cdot P = 0$, hence the term "massless

6·8·6 Transformation of 4 vectors

Example: Observers O and O' are at rest in S and S' where

$$X' = LX, \quad L = \begin{pmatrix} \gamma & -\frac{\gamma v}{c} & & \\ -\frac{\gamma v}{c} & \gamma & & \\ & & 1 & 0 \\ & & 0 & 1 \end{pmatrix}$$

O sees light with frequency ν arriving at angle α to the x axis in the x,y plane. What does O' see? We have $P = \frac{h\nu}{c}\begin{pmatrix} 1 \\ \underline{k} \end{pmatrix}$

$\underline{k} = \begin{pmatrix} \cos\alpha \\ \sin\alpha \\ 0 \end{pmatrix}$, P is the 4-momentum of photons in S. So

$$P' = LP = \begin{pmatrix} \gamma & -\frac{\gamma v}{c} & & \\ -\frac{\gamma v}{c} & \gamma & & \\ & & 1 & 0 \\ & & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ \cos\alpha \\ \sin\alpha \\ 0 \end{pmatrix}\frac{h\nu}{c}$$

$\underline{k}' = \begin{pmatrix} \cos\alpha' \\ \sin\alpha' \\ 0 \end{pmatrix}$

$P'_z = 0$ so we can write $P' = \frac{h\nu'}{c}\begin{pmatrix} 1 \\ \underline{k}' \end{pmatrix}$ for some α', and

$\nu' = \gamma\nu\left(1 - \frac{v}{c}\cos\alpha\right)$ (the Relativistic Doppler Effect)

$\nu'\cos\alpha' = \gamma\nu\left(\cos\alpha - \frac{v}{c}\right)$

$\nu'\sin\alpha' = \nu\sin\alpha$

$\tan\alpha' = \frac{\sin\alpha}{\gamma\left(\cos\alpha - \frac{v}{c}\right)}$ (Stellar Aberration Formula)

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6.9 Newton's 2nd Law in Special Relativity

6.9.1 Momentum Conservation

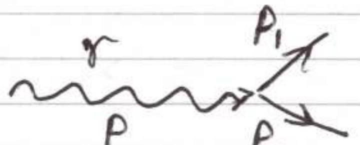
The Lorentz invariant form of Newton's 2nd Law is

4 vector ⇒ $\frac{dP}{d\tau} = F$, the 4-force. For example, the Lorentz force on a charged particle. In the case $F = 0$, P is constant.

For a system of particles (as in chapter 5):

$\sum_i P_i$ = constant i.e E_i = constant, $\underline{p_i}$ = constant

6.9.3 Examples



ii) Photon decay?

Suppose a photon decays into two particles of rest mass m_1 and m_2. $P = P_1 + P_2$ $\quad P \cdot P = 0 \quad P_1 \cdot P_1 = m_1^2 c^2$ $\quad P_2 \cdot P_2 = m_2^2 c^2$

$\Rightarrow P \cdot P = (P_1 + P_2) \cdot (P_1 + P_2) = P_1 \cdot P_1 + P_2 \cdot P_2 + 2P_1 \cdot P_2$

$0 = m_1^2 c^2 + m_2^2 c^2 + 2P_1 \cdot P_2$

Assume $m_1 > 0$. Then, working in the rest frame of particle 1

$P_1 \cdot P_2 = \begin{pmatrix} m_1 c \\ \underline{0} \end{pmatrix} \cdot \begin{pmatrix} E_2/c \\ \underline{p_2} \end{pmatrix} = m_1 E_2 > 0$

which is a contradiction, since each term on the right hand side is greater than or equal to zero. Thus m_1 and m_2 are both zero.

For massless particles, $P_1 = \begin{pmatrix} p_1 \\ \underline{p_1} \end{pmatrix}$, $P_2 = \begin{pmatrix} p_2 \\ \underline{p_2} \end{pmatrix}$

$\Rightarrow 2P_1 \cdot P_2 = 2(p_1 p_2 - \underline{p_1} \cdot \underline{p_2}) = 0$

$\Rightarrow p_1 p_2 (1 - \cos\theta) = 0$, $\cos\theta = 1$, $\theta = 0$

so particles 1 and 2 move together. Essentially, photons do not decay.

iii) Particle Creation

Proton - Proton collisions can result in the formation of an additional proton and an antiproton. → Proton - Antiproton Pair

Proton at rest in lab frame → $P_1 + P_2 = Q_1 + Q_2 + \underbrace{Q_3 + Q_4}$

Incident Proton

In the lab frame: $P_1 = \begin{pmatrix} mc \\ \underline{0} \end{pmatrix}$, $P_2 = \begin{pmatrix} E/c \\ \underline{p} \end{pmatrix}$

Now $Q_1 \cdot Q_2 \geqslant m^2c^2$ because in the rest frame of particle ①

$Q_1 = \begin{pmatrix} mc \\ \underline{0} \end{pmatrix}$, $Q_2 = \begin{pmatrix} E_2/c \\ \underline{p_2} \end{pmatrix}$, $Q_1 \cdot Q_2 = mE_2 = m\underbrace{\sqrt{p_2^2c^2 + m^2c^4}}_{\geqslant m^2c^2}$

Thus $(P_1 + P_2)\cdot(P_1 + P_2) = \underbrace{(Q_1 + \dots + Q_4)\cdot(Q_1 + \dots + Q_4)}_{\geqslant 16m^2c^2}$

$m^2c^2 + m^2c^2 + 2mE \geqslant 16m^2c^2$, $E \geqslant 7mc^2$

This gives the minimum Kinetic Energy (i.e $E - mc^2$) for the creation of the proton antiproton pair.