23/01/12

Fluid Dynamics

What is a fluid? Fluids flow.

Examples: Water, air, syrup, oil. Newtonian Fluids (simple)
- Paint, toothpaste, ketchup, soup, shampoo. Non-Newtonian liquids
- Sand, salt, 3 Gravel-fluids
- Foam

Newtonian Fluids

These have a linear relationship between stress and rate of strain.

Stress - force per unit area (e.g. pressure)

Strain - extension per unit length (e.g. in elasticity)

Rate of strain - rate of extension per unit length

\( \varepsilon = \frac{\Delta l}{l_0} \) (is a gradient of velocity)

In part II, stress and strain rate are considered as tensor quantities, but here, we will consider situations that can be described not using scalar and vector fields.

We shall discuss viscous fluids. However, we shall often make an inviscid approximation.

Pressure is an example of a normal stress. The pressure force per unit area on a surface with normal \( \mathbf{a} \), pointing into the fluid is

\[ F_p = -p \mathbf{a} \text{ surface} \]

Gradients in pressure provide a net force.

High pressure \( \nabla F \) (force per unit volume)
Tangential (Shear) Stress

We find experimentally that the force per unit area required to slide plates at relative speed \( U \) is

\[ \tau_s \propto \frac{U}{h} \]

We write \( \tau_s = \mu \frac{U}{h} \) to define \( \mu \), the dynamic viscosity of the fluid.

Such observations lead to the result that the tangential shear stress exerted by a fluid on a bounding surface with normal \( \mathbf{a} \) pointing into the fluid is

\[ \tau_s = \mu \frac{\partial u}{\partial a} \]

where \( u_n \) is the tangential component of the fluid velocity and \( \tau_s \) is in the direction of \( u_n \).

Steady, parallel, viscous flow

Velocity field \( \mathbf{u} = (u(x), 0) \)

We will start thinking about forces in the \( x \) direction, considering \( p(x) \) and \( p(x + \Delta x) \). We will also have a shear stress, \( \tau_s(y) \) and \( \tau_s(y + \Delta y) \).
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Consider forces acting in the x-direction on the dashed slab exerted by the surrounding fluid. The slab is not accelerating so forces must balance.

\[ p(x) \delta y - p(x + \delta x) \delta y + \frac{\partial}{\partial x} \left( \frac{1}{2} \rho v^2 \right) \delta x = 0 \]

But \( \frac{\partial}{\partial x} = \mu \frac{\partial^2 u}{\partial y^2} \), \( \frac{\partial}{\partial y} = -\mu \frac{\partial^2 v}{\partial x \partial y} \), \( \frac{\partial}{\partial x} = \mu \frac{\partial^2 u}{\partial x^2} \)

\[ \Rightarrow \frac{\partial^2}{\partial x^2} + \mu \frac{\partial^2}{\partial y^2} = 0 \]

Repeat derivation in the y direction:

\[ -\frac{\partial^2}{\partial y^2} = 0 \]

Example Sheet 1: For unsteady parallel viscous flow,

\[ \mathbf{v} = (u(y, t), 0) \] with a body force \( \mathbf{f} \) per unit volume.

\[ \mathbf{F} = (F_x, F_y) \] acting on the fluid then

\[ \rho \frac{\partial^2 u}{\partial x^2} = -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial y^2} + F_x \]

and \( \mathbf{0} = -\frac{\partial p}{\partial y} + F_y \)

Note that in a gravitational field, \( \mathbf{F} = \rho g \)

Boundary Conditions (for viscous fluids)

It has been verified experimentally (e.g., for water down to 2 molecular diameters \( \approx 6\AA \)) that Newtonian fluids satisfy a no-slip condition, that the tangential velocity of the fluid is equal to the tangential velocity of its boundary. For a stationary, rigid boundary \( u_x = 0 \).

Stress condition: Sometimes the tangential stress \( \tau_y \) is prescribed at the boundary rather than the velocity.
If a stress \( \tau \) is applied at a boundary to the fluid then
\[
\frac{\partial^2 u}{\partial y^2} = \tau
\]

**Couette Flow** - driven by boundary stresses or motion.

\[
\text{fluid} \rightarrow u(y) \quad y = \alpha
\]
\[
\text{stationary} \quad y = 0
\]

Steady, and no imposed pressure gradient, so
\[
\frac{\partial^2 u}{\partial y^2} = 0, \quad 0 < y < h
\]
\[
\Rightarrow \quad u = A y + B
\]

\[
\begin{align*}
\partial y^2 = 0 \Rightarrow B = 0, \quad A = \text{const} \\
\frac{u}{y} = \text{linear velocity profile}
\end{align*}
\]

**Poiseuille Flow** - driven by pressure gradient.

\[
\frac{\partial^2 u}{\partial y^2} = -\frac{1}{\mu} \frac{\partial p}{\partial x}
\]

\[
\text{No slip} : \quad u = 0 \quad (0 < y < h)
\]

\[
\begin{align*}
\Rightarrow \quad \rho = \rho g y + f(x) \\
\Rightarrow \quad \frac{\partial^2 u}{\partial y^2} = \frac{1}{\mu} \frac{\partial p}{\partial x} = -f'(y) = -g \quad \text{constant}
\end{align*}
\]

\[
\Rightarrow \quad u = \frac{g}{\mu} y (h-y)
\]

**Derived properties of a flow**

1. **Volume flow** \( q \) : volume of fluid traversing a cross-section per unit time. For parallel flow \( q = \frac{1}{h} \int_{0}^{h} v \, dy \) per unit traverse distance.

\[
\begin{align*}
\text{Couette} : \quad q = \frac{1}{2} u_1 (h) \\
\text{Poiseuille} : \quad q = \frac{1}{2} g h
\end{align*}
\]

2. **Vorticity** \( \omega \) : the curl of the velocity field.

\[
\begin{align*}
\text{Couette flow} : \quad \omega &= \left( 0, 0, -\frac{g}{h} \right) \quad \text{uniform} \\
\text{Poiseuille flow} : \quad \omega &= \left( 0, 0, \frac{g}{h} \right)
\end{align*}
\]
3. Surface Stress

\( \tau_s \) - tangential force per unit area exerted by fluid

\( \tau_s = \mu \frac{\partial u}{\partial n} \), \( n \) points into the fluid

**Couette Flow**

\[ \tau_s = \begin{cases} \mu \frac{1}{h} \frac{h}{4} & \text{for } \frac{y}{h} = 0 \\ \mu \frac{1}{h} \frac{h}{4} & \text{for } \frac{y}{h} = 1 \end{cases} \]

**Poiseuille Flow**

\[ \tau_s = \begin{cases} \frac{\mu}{4} \frac{h^2}{y} & \text{for } \frac{y}{h} = 0 \\ \frac{\mu}{4} \frac{h^2}{y} & \text{for } \frac{y}{h} = 1 \end{cases} \]

Note that the Poiseuille flow has \( \tau_s \) independent of \( \mu \).
Corette - Poiseuille Mixture

Pressure gradients may develop to enforce incompressibility.

Assume that the air exerts no tangential stress on water. Make tangential stress on water: $\tau = -\rho g \cos \alpha (y-h) + P_0$, $\tau = -\rho g \cos \alpha (y-h)$.

\[ \frac{\partial P}{\partial y} = -\rho g \cos \alpha \]

\[ \frac{\partial P}{\partial x} = -\rho g \sin \alpha \]

Boundary Conditions:

- No slip: $u(x, 0) = 0$
- No tangential stress: $\mu \frac{\partial u}{\partial y} (0, h) = 0$

\[ u(x, y) = \frac{\rho g}{2\nu} \sin \alpha y \]

\[ \frac{\partial u}{\partial y} (2h, y) = \frac{\rho g}{2\nu} \sin \alpha y (2h-y) \]

where $\nu = \frac{\mu}{\rho}$ is the kinematic viscosity of the water.

Unsteady Parallel Viscous Flow

Consider a semi-infinite domain $y > 0$ initially at rest. At time $t = 0$ the boundary $y = 0$ is set into motion with speed $U$.

\[ \frac{\partial u}{\partial y} = \rho g, \quad \frac{\partial u}{\partial x} = \rho g y + f(x) \]

\[ \frac{\partial u}{\partial x} = \frac{\rho g}{\sigma^2} \]

\[ u(x, \infty) = U \quad \text{for} \quad t > 0, \quad u = 0 \quad \text{as} \quad y \to \infty, \quad u(0, t) = 0 \]

The velocity satisfies the diffusion equation and the kinematic viscosity $\nu = \frac{\mu}{\rho}$ can be thought of as a diffusivity for momentum (or viscosity, see later).
The diffusion equation can be solved by Fourier transform in time. 

\[
\text{Fourier series in } t, \quad (\text{see separation of variables, Laplace transforms, etc.)}
\]

Similarity solution: see III. Differential Equations.

In the infinite domain, the diffusion equation has a similarity solution:

\[
x(t) = \left(\frac{\xi}{t}\right)^{1/2} \quad t = \frac{1}{\beta^2}
\]

\[
\frac{\partial x}{\partial t} + \frac{1}{t} \frac{\partial x}{\partial \xi} = 0, \quad f = \exp\left(-\frac{t}{\xi^2}\right), \quad \alpha = \frac{1}{\beta^2}
\]

Kinematic vs. Dynamic Viscosity

\[
\eta_t \approx \frac{10^{-3}}{10^3}
\]

Water: \(\frac{1.00 \times 10^{-3}}{0.001}\) Water \(\text{rot.} \text{pt. } 20^\circ\)

Air: \(2 \times 10^{-5}\)

i) \(\text{V} = 20 \text{ ft/sec} \) of water, so the motion is reduced further into the air.

ii) \(\text{Kinematic Stress on } y = 0 \text{ is } T_y = \mu \frac{\partial u}{\partial y} = \mu \frac{1}{2} \left(-\frac{1}{t}\right) \frac{\partial x}{\partial t} = \mu \frac{1}{2} \frac{1}{\beta^2} = \mu \frac{1}{\beta^2} \quad t = \frac{1}{\beta^2}
\]

\[
\text{(Area of water) } \times 1 \text{ (SI units) }, \quad \text{(Area of air) } \times 1.6 \times 10^{-3}
\]

So water exert a much greater shear force on the boundary for the same boundary motion.

Dimensional Analysis

Covaring equation \(\frac{v}{\text{rot. pt.}} = \frac{1}{2} \frac{\alpha}{\beta^2} = \frac{1}{2} \frac{1}{\beta^2} \). Let \(\beta^2\) be a dimensionless quantity.

(justify)

Note that the current problem has no intrinsic length scale.

There is a balance between inertia and viscous dissipation when
 Fluid Dynamics

\[ \rho \frac{u}{t} \sim \mu \frac{u}{x} \Rightarrow \frac{u}{x} \sim \sqrt{t} \]

We can thus determine characteristic scales of the flow without solving the differential equation.
Fluid Dynamics

Characterization/Visualization of Flow

Lagrangian Picture
Mark (dyne) a fluid particle and follow its trajectory. Trajectories can cross and it is difficult to formulate differential equations. Properties (e.g., density) would be written as $\rho(t, x_0)$ for a particle at time $t$ that was released from $x_0$ at time $t_0 = 0$ say.

Eulerian Picture
Sit still and watch the world go by. Write all dependent variables (e.g., $u$, as functions of fixed location $x$ at time $t$.

Material time derivative (Eulerian Picture)
Consider a time dependent field $f(x, t)$

Along a path $x = x(t)$
$$\frac{df}{dt}(x(t), t) = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial x} \frac{dx}{dt} = x \cdot \nabla f + \frac{\partial f}{\partial x}$$

If $x(t)$ is the Lagrangian path followed by a fluid particle, then $x(t) = x$, the local fluid velocity. We write $\frac{df}{dt} = \frac{df}{dt}$ for the time derivative along a path (trajectory) of a fluid particle.

Material time derivative: $\frac{df}{dt} = \frac{\partial f}{\partial x} x \cdot \nabla f \leq \text{Advection time derivative}$

Lagrangian time derivative: Eulerian time derivative

Conservation of Mass
Consider a fixed region of space $D$ with boundary $\partial D$ and outward normal $n$.
Mass is not created or destroyed so the mass inside $D$ can only change by a net flow across $\partial D$

\[
\frac{d}{dt} \int_D \rho \, dV = - \int_{\partial D} \rho \mathbf{n} \cdot \mathbf{a} \, dS \quad \Rightarrow \quad \int_{\partial D} \mathbf{n} \cdot (\rho \mathbf{a}) \, dS = \oint \frac{\partial}{\partial t} \mathbf{v} \cdot d\mathbf{S}
\]

using the divergence theorem.

\[
\Rightarrow \int_{\partial D} \mathbf{n} \cdot (\rho \mathbf{a}) \, dS = \oint \mathbf{v} \cdot (\rho \mathbf{v}) \, dV = 0
\]

This is true for arbitrary $D$ as $\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \rho = 0$ pointwise.

This has the general form of a conservation law:

rate of change of stuff + divergence of the stuff flux = 0

**Mass flux**

\[ \rho \mathbf{v} \]

**Product rule**

\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \]

\[ \frac{\partial \rho}{\partial t} + \rho \nabla \cdot \mathbf{v} = 0 \]

If the fluid is incompressible then the density of a fluid particle cannot change.

i.e.

\[ \frac{\partial \rho}{\partial t} = 0 \quad \Rightarrow \quad \nabla \cdot \mathbf{v} = 0 \quad \text{continuity equation} \]

Note that in parallel flow, $\mathbf{v} = (v, 0, 0) \quad \Rightarrow \quad \nabla \cdot \mathbf{v} = \frac{\partial v}{\partial x} = 0$

N.B. an unconfined fluid can be treated as incompressible provided $|v| \ll \mathbf{C}$, speed of sound.  
Cair $= 340 \text{ m/s}$, Carbon $= 1500 \text{ m/s}$

**Kinematic Boundary Condition**

Consider a material boundary moving with velocity $\mathbf{U}$.  
Fluid

In a local frame of reference moving with velocity $\mathbf{U}$, the fluid velocity has relative value $\mathbf{v}' = \mathbf{v} - \mathbf{U}$ and the boundary is stationary.

\[ \mathbf{U} \cdot \mathbf{n} = 0 \quad \text{Fluid cannot cross the boundary so } \mathbf{v} \cdot \mathbf{n}' = 0 \]

\[ \Rightarrow \quad \mathbf{n} \cdot \mathbf{v} = \mathbf{n} \cdot \mathbf{U} \quad \text{at boundary}. \]
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1. At a rigid boundary, \( \mathbf{u} = 0 \) \( \Rightarrow \nabla \cdot \mathbf{u} = 0 \)

2. At a free material boundary, such as the surface of a water wave,

Think of the surface as a contour:

\[ F(x, y, z, t) = z - \frac{1}{3} \varepsilon(x, y, t) \quad ( \text{zero at a contour} ) \]

The normal to the surface is the normal to the contour of \( F \).

\[ \mathbf{n} = \nabla F = \left( -\frac{\partial F}{\partial x}, -\frac{\partial F}{\partial y}, 1 \right) \]

\[ \mathbf{n} = (0, 0, 1) \quad , \quad \mathbf{w} = (u, v, w) \]

\[ \Rightarrow u \frac{\partial F}{\partial x} + v \frac{\partial F}{\partial y} + w = \frac{\partial F}{\partial z} \]

\[ \Rightarrow w = \frac{1}{\varepsilon} \frac{\partial \varepsilon}{\partial z} + \nu \frac{\partial^2 \varepsilon}{\partial x^2} + \nu \frac{\partial^2 \varepsilon}{\partial y^2} \]

\[ w = \frac{1}{\varepsilon} \]
Stream functions for 2-D incompressible flow

Incompressible flow \( \Rightarrow \nabla \cdot \mathbf{u} = 0 \) \( \Rightarrow \mathbf{u} = \nabla \times \mathbf{A} \)

for some vector potential \( \mathbf{A} \).

If \( \mathbf{u} = (u, v, 0) \) in Cartesian components, we can find \( \mathbf{A} = (0, 0, \Psi(x, y)) \)

Note: \( u = u(x, y), \quad v = v(x, y) \)

Putting this together: \( \Psi = \left( \frac{\partial u}{\partial y}, -\frac{\partial v}{\partial x}, 0 \right) \)

The scalar function \( \Psi(x, y) \) is called the stream function for the flow.

Properties of the stream function

The contours of \( \Psi(x, y) \) have normals \( \mathbf{n} = \nabla \Psi = \left( \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, 0 \right) \)

So \( \mathbf{n} \cdot \mathbf{u} = \frac{\partial u}{\partial x} u + \frac{\partial u}{\partial y} v = 0 \)

So \( \mathbf{u} \) is orthogonal to the normals of the contours of \( \Psi \).

\( \Rightarrow \mathbf{u} \) is tangent to the contours of \( \Psi \).

(i) The contours of \( \Psi \) are streamlines.

N.B. If the flow is unsteady then streamlines are not particle paths.

The way to visualise streamlines is to seed the fluid with lots of particles and take a photograph with an open shutter of short duration.

e.g. \( \mathbf{u} = (t, t, 0) \)

Lagrangian particle paths are given:

\[ t = 0 \quad \Rightarrow \quad \mathbf{x}(0) = \mathbf{x}_0 \]

\[ t = 1 \quad \Rightarrow \quad \mathbf{x}(1) = \mathbf{x}_0 \]

\[ \mathbf{x} = t, \quad \mathbf{y} = 1, \quad \mathbf{u} = \frac{1}{2} t^2 + \mathbf{x}_0, \quad \mathbf{y} = t + \mathbf{y}_0 \]

where \( t \) can be considered as a parameter for the path. Particle paths can be found by eliminating \( t \)

\[ (\mathbf{x} - \mathbf{x}_0) = \frac{1}{2} (\mathbf{y} - \mathbf{y}_0)^2 \]
(ii) The flow is faster where streamlines are closer.

(iii) Volume flux (per unit length in the z direction) crossing any curve from $x_0$ to $x_1$

\[ N_1 = N(x_1), \quad N_0 = N(x_0) \]

(iv) $N$ is constant on a stationary, rigid boundary. We are always free to choose $N = 0$ on one of the stationary boundaries of a flow.

Plane-polar coordinates ($r, \theta$)

For 2-D flow expressed in plane polar, embedded in cylindrical polar ($r, \theta, z$) and write $u = \nabla \times (0, 0, N) = \nabla \times \left( \frac{a_r}{a^2}, 0, 0, \frac{a_r}{a^2}, 0, 0 \right)$

Navier-Stokes Equation

Newton's 2nd law for a fluid particle is

\[ \rho \frac{\partial \mathbf{u}}{\partial t} = -\nabla p + \mu \nabla^2 \mathbf{u} + \mathbf{f} \]

maxx x acceleration = surface forces + body forces

Note that the acceleration is Lagrangian (material) derivative of the velocity.

Derivation of the viscous term $\mu \nabla^2 \mathbf{u}$ requires consideration of the stress tensor.

see part II fluids.

[Note: $\nabla^2 \mathbf{u} = \nabla (\nabla \cdot \mathbf{u}) - \nabla \times (\nabla \times \mathbf{u})$]

$= -\nabla \times (\nabla \times \mathbf{u})$ for incompressible flow.

In Cartesian we are straightforwardly $\nabla^2 \mathbf{u} = (\nabla^2 u_x, \nabla^2 u_y, \nabla^2 u_z)$
Exercise. Show that if \( \mathbf{u} = (u_1, u_2, 0, 0) \) then the Navier-Stokes equations reduce to the earlier parallel flow equations derived.

In a gravitational field, \( \mathbf{E} = \rho \mathbf{g} \)

Navier-Stokes equations:

\[
\rho \left( \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\nabla p + \mu \nabla^2 \mathbf{u} + \mathbf{E}
\]

The term \( \mathbf{u} \cdot \nabla \mathbf{u} \) is strongly non-linear and gives rise to all the interesting behaviour of fluids.
Hydrostatic Pressure

If φ = 0, write \( p = \rho h \)

\[ 0 = -\nabla \rho h + \rho g \]

\[ \Rightarrow \rho h = \rho g x + \rho_0 = \rho_0 - \rho g x \]

Pressure decreases upwards.

Archimedes' Static Force on a submerged body

By the divergence theorem

Fluid

\[ E = \int \rho \Delta \, ds = - \int \omega \Delta \, dV \]

Density

\[ \rho = \rho_0 \frac{dV}{dV} = -\rho g \int dV = -\rho g V = -M g \]

where \( M = \rho g \) is the mass of fluid displaced by the body.

Upthrust = weight of fluid displaced.

**Dynamic Pressure** - causing or resulting from fluid flow.

Write \( p = \rho h + p' \) dynamic pressure

\[ \Rightarrow \rho \frac{dV}{dt} = -\nabla p' + \mu V^2 \]

1. We usually drop the prime so that \( p \) is referred to as the dynamic pressure.

2. Every fluid particle is neutrally buoyant.

3. If there is a free surface (e.g., water/air) then we do need to consider gravity.

4. The dynamic pressure is often determined internally and is always sufficient to maintain the incompressibility constraint; \( \nabla \cdot \mathbf{u} = 0 \)

Reynolds Number

Suppose the flow has a characteristic (typical) magnitude \( U \) and

intrinsic length scale, externally imposed by geometry.

These two scales define a timescale \( T = \frac{L}{U} \).
Suppose that pressure differences have characteristic magnitude $P$. What is the relative importance of the different terms in the Navier-Stokes equations?

\[
\frac{\partial u}{\partial t} + u \cdot \nabla u = -\frac{1}{\rho} \nabla P + \nu \nabla^2 u
\]

\[
\frac{u}{\mu} : U \frac{u}{L} \approx \frac{1}{\rho} \frac{P}{L} : \nu \frac{u}{L^2}
\]

\[
1 : 1 : \frac{1}{\rho \frac{u}{L}} : \nu \frac{u}{L^2} = \frac{1}{Re}
\]

The Reynolds number $Re = \frac{u L}{\nu}$ gives the relative magnitude of the inertial terms to the viscous terms.

N.B. Pressure must always scale to balance the dominant term in the equation so that we can impose $\nabla \cdot u = 0$.

Small $Re$, $Re \ll 1$

Small lengths (e.g. cells), Slow flows (slow running tap)

Large viscosity (e.g. lava, oil)

Inertial terms are negligible $P = \rho u \frac{u}{L} = \rho \frac{u^2}{L}$

We can approximate by the Stokes Equations $0 = -\nabla P + \mu \nabla^2 u$ and $\nabla \cdot u = 0$. Note that $\nu \ll P$

These will be studied in detail next year.

Large $Re$, $Re \gg 1$

Viscous terms are negligible on extrinsic length scales. $P = \rho u$, scales with the momentum flux. On extrinsic length scales, we can approximate the NS equations with the Euler equations: $P \frac{\partial u}{\partial t} = -\nabla P$, $\nabla \cdot u = 0$

Note that pressure gradients give rise to accelerations.
Intrinsic length scales on which inertial terms balance viscous terms is
\[ S \text{ where } \frac{\nu^2}{S} = \nu \frac{U}{S} \Rightarrow \frac{S}{U} = \frac{1}{\nu \text{Re}} < 1 \]

At large Reynolds numbers, viscosity acts on small length scales
(e.g. rigid boundaries)

\[ \nabla \cdot \mathbf{u} = \sum_{i=1}^{3} \frac{\partial (u_i)}{\partial x_i} = u_1 \frac{\partial u_1}{\partial x_1} + \cdots + u_3 \frac{\partial u_3}{\partial x_3} \]
\[ \rho \left( \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\nabla p + \mu \nabla^2 \mathbf{u} \]

At large \( Re \), viscous terms are unimportant on the (extrinsic) length scales of the flow.

\[ \text{A Case Study - Stagnation Point Flow} \]

There is an exact solution of NS in which \( \nabla \cdot \mathbf{u} = 0 \) and that the stream function \( \psi = \text{const} \).

The solution has the form \( \mathbf{u} = (\text{Ex} g'(\eta), -\text{Ex} g(\eta), 0) \) where \( \eta = \frac{y}{\text{Ex}} = \frac{\text{Ex}}{\text{L}} y \). The (intrinsic) length scale \( \varepsilon = \frac{\text{L}}{\text{Ex}} \).

Exercise

i) Show that \( \nabla \cdot \mathbf{u} = 0 \) and that the stream function \( \psi = \text{Ex} g(\eta) \) with \( \psi = \text{Ex} g \) as \( y \to \infty \).

ii) Show that the NS equations give

\[ \text{Ex} \left( g'' - g g'' \right) = -\frac{1}{\rho} p_x + \text{Ex}^2 g'' \]

\[ \text{Ex} \left( g g'' \right) = -\frac{1}{\rho} p_y - \text{Ex} \text{Ex} g'' \]

Differentiate:

\[ \frac{\partial g'}{\partial x} \Rightarrow p_{xy} = 0 \]

\[ \frac{\partial g}{\partial y} \Rightarrow (g'' - g g'')' = g'' \]

iii) Show that the boundary conditions give

\[ g' \to 1, \quad g \to \text{const} \quad \text{as} \ \eta \to \infty, \quad g = g' = 0 \quad \text{when} \ \eta = 0 \]

All dimensional parameters have been absorbed into the scaled variables \( g \) and \( \eta \).
So we only need to solve the ODE once to find 

\[ \mathbf{v} = (Ex, -Ey, 0) \]

The far field velocity \( \mathbf{v} = (Ex, -Ey, 0) \) is reached to an excellent approximation when \( Re \gg 1 \Rightarrow y \gg 8 = \frac{L}{D} \).

Recall that \( \mathbf{u}_\infty = (Ex, -Ey, 0) \) in the far field.

Overall Picture

If we're interested in the flow on scales much larger than \( \delta \), then solve the equations in \( y > \delta \), ignoring the viscous terms.

Formally, we will take the limit \( \delta \to 0 \) and solve the Euler equations in \( y > 0 \). BUT we can't then use the no-slip condition.

Inviscid Approximation

If \( Re \gg 1 \) then solve the Euler equations

\[
\frac{D \mathbf{u}}{Dt} = -\nabla p \quad (+f)
\]

\[
\nabla \cdot \mathbf{u} = 0, \quad \mathbf{u} \cdot \mathbf{n} = 0 \quad \text{at stationary, rigid boundaries}.
\]

The no-slip condition cannot be imposed on solutions of the Euler equation.

Exercise: Show that the flow

\[ \mathbf{u} = (Ex, -Ey, 0) \] satisfies the Euler equations in \( y > 0 \) with a rigid boundary at \( y = 0 \) and a pressure field

\[
p = p_\infty - \frac{\rho E^2}{2} (x^2 + y^2)
\]

The pressure field acts as an internal reaction force impinging the constraint of incompressibility.
Momentum Equation

For incompressible, inviscid flow. The momentum of fluid inside a region $D$ with boundary $\partial D$ can change owing to:

1. Momentum flowing across the boundary.
2. Volume (body) forces.
3. Surface pressure forces.

\[
\frac{d}{dt} \int_D \rho u \, dV = -\int_{\partial D} \rho u \cdot n \, ds + \int_D f_i \, dV - \int_{\partial D} \rho n_i \, ds
\]

In components:

\[
\frac{d}{dt} \int_D \rho u_i \, dV = -\int_{\partial D} \rho u_i n_i \, ds + \int_D f_i \, dV - \int_{\partial D} \rho n_i \, ds
\]

\[
\Rightarrow \int_D \rho \frac{\partial u_i}{\partial t} + \rho \frac{\partial}{\partial x_j} (u_i u_j) \, dV = \int_D \frac{\partial}{\partial x_i} \left( \rho f_i \right) \, dV
\]

This is true for arbitrary region $D$, so

\[
\rho \frac{\partial u_i}{\partial t} + \rho \left( u_j \frac{\partial u_i}{\partial x_j} + u_i \frac{\partial u_j}{\partial x_j} \right) = -\frac{\partial}{\partial x_i} \left( \rho f_i \right)
\]

\[
\Rightarrow \rho \frac{\partial u_i}{\partial t} = -\nabla \rho + \nabla X
\]

Euler Momentum Equation

\[
\frac{\partial u_i}{\partial t} = -\nabla \rho + \nabla X
\]

The acceleration of a fluid particle, $\frac{\partial u_i}{\partial t}$, is the local rate of change of the velocity at a point.

Conservative body forces

\[
\mathbf{f} = -\nabla X, \quad \text{e.g., gravity}, \quad \mathbf{f} = \rho g = \nabla (\rho g \cdot z), \quad X = -\rho g z
\]

Momentum Integral for steady flow

From (X),

\[
\sigma = -\int_{\partial D} \rho u \cdot (u \cdot n) \, ds - \int_D \nabla X \cdot dV = \int_{\partial D} \rho n \, ds
\]

\[
\Rightarrow \int_{\partial D} \rho u \cdot (u \cdot n) + \rho n + X \, ds = 0
\]
Application of Momentum Integral

Curved hose-pipe

Steady flow. Conservation of mass: $\frac{du}{dt} = \text{constant} = u$

Neglect gravity. No acceleration along the pipe: $\Rightarrow p$ is constant

$$\int \text{Curved surface} \rho \frac{d\mathbf{u}}{dt} \cdot \mathbf{n} \, dS + A \rho (-\mathbf{u} \times \nabla) \cdot \mathbf{u} + A \rho \frac{d\rho}{dt} \frac{u}{u^2} + A \rho_2 \frac{1}{3} \nabla \cdot \mathbf{u} = 0$$

Force on a pipe:

$$E = \int p \rho \, dS = -A \left( \rho \mathbf{u}^2 + p \right) (n_1 + n_2)$$

Bernoulli's Equation

Vector identity: $\mathbf{u} \times \left( \nabla \times \mathbf{u} \right) = \nabla \left( \frac{1}{2} \rho \mathbf{u}^2 \right) - (\mathbf{u} \cdot \nabla) \mathbf{u}$

From Euler's Equation: $\rho \frac{\partial \mathbf{u}}{\partial t} + \nabla \left( \frac{1}{2} \rho \mathbf{u}^2 \right) - \rho \mathbf{u} \times \left( \nabla \times \mathbf{u} \right) = -\nabla p - \nabla \mathbf{X}$

Dot with $\mathbf{u}$:

1. Steady Flow $\mathbf{u} \cdot \nabla \left( \frac{1}{2} \rho \mathbf{u}^2 + p + \mathbf{X} \right) = 0$

Directional derivative with respect to time along a streamline

$\Rightarrow \mathbf{H} = \frac{1}{2} \rho \mathbf{u}^2 + p + \mathbf{X} = \text{constant along streamline}$

Bernoulli's Theorem.

Use the divergence theorem recalling $\nabla \cdot \mathbf{u} = 0$

2. Integrate: $\frac{d}{dt} \int \frac{1}{2} \rho \mathbf{u}^2 \, dV = -\int \mathbf{H} \cdot \mathbf{n} \, dS$

So the rate of change of the kinetic energy inside $V$ is equal to the flux of $\mathbf{H}$ across the boundary of $V$. So $H$ is the transportable energy of the fluid.

Application of Bernoulli's Equation

i) Pitot tube for measuring air speed.

Along stagnation point streamline

$$\frac{1}{2} \rho \mathbf{u}^2 + p_0 = p_1 \Rightarrow u = \left( \frac{p_1 - p_0}{\rho} \right)^{1/2}$$

Measuring $p_1$ gives $u$. In the barometer $p_1 - p_0 = \rho w \, gh$

$$u = \frac{1}{2} \rho w \, gh^{1/2}$$
Boussinesq's Theorem

\[ H = \frac{1}{2} \rho \left( u_1^2 + u_2^2 \right) + p + \chi = \text{constant along streamlines} \]

1) Measuring flow rate in a pipe

Area

\[ A_1 \rightarrow A_2 \]

\[ \rho \frac{u_1^2}{A_1} + p_1 = \rho \frac{u_2^2}{A_2} + p_2 \]

\[ \frac{1}{2} \rho \left( \frac{u_1^2}{A_1} \right) + p_1 = \frac{1}{2} \rho \left( \frac{u_2^2}{A_2} \right) + p_2 \]

\[ \Rightarrow \left( \frac{A_2}{A_2 - A_1} \right) \rho \frac{u_1^2}{A_1} = \frac{1}{2} \rho \left( p_1 - p_2 \right) \]

\[ \Rightarrow \rho \frac{u_1^2}{A_1} = \frac{1}{2} \rho g h \]

\[ \frac{A_1}{A_2 - A_1} \]

\[ \frac{A_2}{A_2 - A_1} \]

Linear flow

Consider the flow in the neighborhood of a fixed point \( x_0 \)

\[ u(x) = u_0 + (x - x_0) \cdot \nabla u(x_0) + \ldots \]

\[ \approx u_0 + (x - x_0) \cdot \nabla u \]

- linear approximation

\[ \nabla u = \frac{\partial u_i}{\partial x_j} = E_{ij} + \Omega_{ij} = E + \Omega \]

\[ E_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right), \text{ symmetric} \]

\[ \Omega_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \right), \text{ antisymmetric} \]

Recall: vorticity \( \omega = \nabla \times u \)

Note: \( \omega \times r = (\nabla \times u) \times r \]

\[ (\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}) r_j = Z \Omega_{ij} r_j = 2 \Omega \times r \]

Therefore

\[ u \approx u_0 + E \cdot r \]

\[ + \frac{1}{2} \omega \times r \]

Uniform Flow

Pure Strain

Pure Rotation

Note that the local rotation rate = \( \frac{1}{2} \) the vorticity

Note also that the strain rate tensor \( E \) is symmetric, and hence diagonalized with orthogonal eigenvectors, and \( E \) is traceless, because \( \nabla \cdot u = 0 \)

\[ E_{ik} = \frac{\partial u_i}{\partial x_k} = \nabla_{ik} u = 0 \]

With respect to principle axes, \( E = (E_1, E_2, E_3) \) where \( E_1 + E_2 + E_3 = 0 \)
Vorticity Equation

The Navier–Stokes equations for a viscous fluid give:
\[ \rho \left( \frac{\partial \mathbf{u}}{\partial t} + \nabla \times \mathbf{u} \right) = -\nabla p - \nabla \cdot \mathbf{F} + \mu \nabla^2 \mathbf{u} \]
\[ \Rightarrow \rho \left( \frac{\partial \mathbf{u}}{\partial t} + \nabla \times \left( \frac{1}{2} \mathbf{u} \cdot \mathbf{u} \right) - \mathbf{u} \times \mathbf{\omega} \right) = -\nabla p - \nabla \cdot \mathbf{F} + \mu \nabla^2 \mathbf{u} \]

Take the curl:
\[ \frac{\partial \mathbf{\omega}}{\partial t} - \nabla \times \left( \mathbf{u} \times \mathbf{\omega} \right) = \nabla^2 \mathbf{\omega} \]

But
\[ \nabla \times \left( \mathbf{u} \times \mathbf{\omega} \right) = \nabla \left( \mathbf{u} \cdot \mathbf{\omega} \right) - \left( \mathbf{\omega} \cdot \nabla \right) \mathbf{u} - \left( \mathbf{\omega} \cdot \nabla \right) \mathbf{u} + \left( \mathbf{\omega} \cdot \nabla \right) \mathbf{u} \]

So
\[ \frac{\partial \mathbf{\omega}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{\omega} = \nabla \left( \mathbf{u} \cdot \mathbf{\omega} \right) + \nabla \left( \mathbf{\omega} \cdot \nabla \right) \mathbf{u} \]

\[ \frac{\partial \mathbf{\omega}}{\partial t} \text{ is the local rate of change of vorticity} \]
\[ \mathbf{u} \cdot \nabla \mathbf{\omega} \text{ represents advection of vorticity} \]
\[ \mathbf{\omega} \cdot \nabla \mathbf{\omega} \text{ represents amplification (or reduction) of vorticity by stretching (or compression)} \]
\[ \nabla^2 \mathbf{\omega} \text{ represents dissipation of vorticity by the action of viscosity and also allows for the generation of vorticity by the no-slip condition at rigid boundaries.} \]

Vortex amplification by stretching

The vorticity equation in an inviscid fluid is:
\[ \frac{\partial \mathbf{\omega}}{\partial t} = \mathbf{\omega} \cdot \nabla \mathbf{\omega} \]
\[ \Rightarrow \frac{\partial}{\partial t} \left( \frac{1}{2} \mathbf{\omega}^2 \right) = \mathbf{\omega} \cdot \left( \mathbf{E} + \frac{\mathbf{\omega}}{2} \right) \cdot \mathbf{\omega} = \mathbf{\omega} \cdot \mathbf{E} \cdot \mathbf{\omega} \]

wrt principal axes of \( \mathbf{E} \):
\[ \frac{\partial}{\partial t} \left( \frac{1}{2} \mathbf{\omega}^2 \right) = \mathbf{E} \cdot \mathbf{\omega} \cdot \mathbf{\omega} \]

Suppose \( \mathbf{\omega} = (\omega_1, 0, 0) \), magnitude of \( \mathbf{\omega} \) depends on the sign of \( \mathbf{E} \).

Then \( \frac{\partial}{\partial t} \left( \frac{1}{2} \mathbf{\omega}^2 \right) = \mathbf{E} \cdot \mathbf{\omega} \cdot \mathbf{\omega} \) and the vorticity grows or decays exponentially depending on the sign of \( \mathbf{E} \).

Note that the fluid associated with \( \mathbf{E} \) is \( \mathbf{u} = (E_x, E_y, E_z) \).

So \( \mathbf{E} > 0 \) corresponds to stretching parallel to the \( x \)-axis.

E.g. hurricane. The hurricane circulates cyclonically (in the direction of Earth's rotation).

\[ \text{Amplification of planetary vorticity} \]

[Diagram of planetary vorticity]
Fluid Dynamics

Vorticity Equation
\[ \frac{\partial \omega}{\partial x} = \omega \cdot \nabla \times \mathbf{u} = \nabla \times \left( \frac{1}{2} \mathbf{u} \times \mathbf{u} \right) \]

Vortex Stretching
Considering two neighboring (Lagrangian) fluid particles:
\[ \frac{\partial \mathbf{x}_1}{\partial t} = \mathbf{u} (\mathbf{x}_1) \quad \frac{\partial \mathbf{x}_2}{\partial t} = \mathbf{u} (\mathbf{x}_2) \]
\[ \implies \frac{\partial}{\partial t} \mathbf{S}_t = \mathbf{u} (\mathbf{x}_2) - \mathbf{u} (\mathbf{x}_1) = \mathbf{u} (\mathbf{x}_1) + \mathbf{S}_t \cdot \nabla \mathbf{u} (\mathbf{x}_1) + \ldots - \mathbf{u} (\mathbf{x}_1) \]
\[ = \mathbf{S}_t \cdot \nabla \mathbf{u} \quad \text{to leading order} \]
\[ \mathbf{S}_t \text{ satisfies the same equation as } \mathbf{u}, \text{ so vorticity is stretched as fluid elements are stretched.} \]

Inviscid (homogeneous) fluid
\[ \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\nabla p + \nabla \times \mathbf{F} \]

Vorticity can be amplified, advected, but cannot be generated.

Aside
i) Vorticity is generated by viscous action near rigid boundaries.
ii) Vorticity can also be generated by homogeneity.

Inviscid, irrotational flow
If \( \nabla \times \mathbf{u} = 0 \) at \( t = 0 \), and the fluid is inviscid, then \( \nabla \times \mathbf{u} = 0 \) for all time.
\[ \mathbf{u} = \nabla \phi \text{ for some scalar "velocity potential"} \]

Incompressibility: \( \nabla \cdot \mathbf{u} = 0 \implies \nabla^2 \phi = 0 \), Laplace's Equation

Three-dimensional flows
i) Spherically symmetric flows:
\[ \frac{\partial}{\partial r} \left( r^2 \frac{\partial \phi}{\partial r} \right) = 0 \]
Velocity field \( \mathbf{u} = \nabla \phi = \frac{\partial \phi}{\partial r} \mathbf{e}_r = \frac{A}{r} \mathbf{e}_r \)
\[ (B = 0 \quad \text{where?}) \]
Volume flux across the surface of the sphere \( r = a \) is\[ q_r = \int \vec{a} \cdot \vec{dS} = \int \frac{\vec{a}}{a^2} \cdot \vec{dS} = \int \frac{\vec{a}}{a^2} \cdot \frac{A}{2\pi r^2} \cdot r \cdot 2\pi r = \frac{A}{a^2} + 4\pi a^2 = 4\pi a^2\]

Representing the point source of fluid of strength (volume flux) \( q_r \) at the origin Note: \( \nabla^2 q_r = q_r \delta(\vec{r}) \), \( \delta \) is the 3D Dirac delta function for point fluid.

ii) Axisymmetric Solution

In spherical polar coordinates \( \nabla^2 f = g = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial f}{\partial r} \right) \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial f}{\partial \theta} \right) \]

\[ f = \frac{1}{n+1} (A_n r^n + B_n r^{-n-1}) P_n(\cos \theta) \quad \text{with} \quad P_n \text{ the } n\text{th Legendre polynomial}\]

Note: uniform flow past a sphere.

1. \( \nabla^2 f = 0 \) \( \theta = \pi \)
2. \( f \sim UC \cos \theta \) \( \theta \to 0 \)
3. \( u_r = \frac{\partial f}{\partial r} = 0 \) at \( r = a \) (fluid cannot penetrate rigid boundary)

Notes: equation is linear 

Boiling \( UC \cos \theta \) \( (= P_n(\cos \theta)) \)

\( P_n(\cos \theta) \) \( \text{is an eigenfunction of} \quad \nabla^2 \text{ \( \text{in spherical space.} \) So the solution is} \]

\[ f = \frac{1}{n+1} (A_n r^n + B_n r^{-n-1}) P_n(\cos \theta) \]

So \( A = \frac{UC}{\sin \theta} \)

\[ f = \frac{UC}{\sin \theta} (r + \frac{3}{2} \cos \theta) \]

uniform flow \( \to \) dipole field

Velocity and Pressure

\[ u_r = \frac{\partial f}{\partial r} = UC \left( 1 - \frac{a^2}{r^2} \right) \cos \theta \]

\[ u_\theta = -\frac{\partial f}{\partial \theta} = UC \left( 1 + \frac{a^2}{r^2} \right) \sin \theta \]

Note that \( u_\theta = 0 \) on \( r = a \).

Use Bernoulli's Theorem on the stagnation point streamline

\[ \frac{1}{2} \rho \frac{U^2}{r^2} + P_a = \rho + \frac{1}{2} \rho U^2 \frac{1}{4} \sin^2 \theta \quad \rho = \rho_a + \frac{1}{2} \rho U^2 \left( \frac{1}{a} \right)^2 \]

Note: \( P_a = \rho a^2 \) \( = \rho a^2 + \frac{1}{2} \rho U^2 \) \( \text{in the sphere} \)

Pressure is symmetric first and after (around the equator) so the net force on the sphere is zero! (D'Alembert's Paradox)
Fluid Dynamics

Solid Sphere (non-examinable)

Empirically, \[ F = \rho U^2 \pi a^2 \frac{x}{2} C_D, \] where \( C_D \) is a measured drag coefficient.

\( C_D = C_D(Re) \approx 0.4 \) for large \( Re \).

Bubble

Potential flow solution is reasonable in several circumstances.

Exercise: K.E. of fluid = \( \frac{1}{2} \rho U^2 a^2 \left( \frac{dV}{dt} \right) = \frac{1}{2} M_U U^2 \)

where \( M_U \) is the "added mass" (better terminology might be "added inertia").

\[ \frac{1}{2} M_U U^2 = \frac{1}{2} M_U U^2 - \int M_0 \, dl \] (missing mass)

\[ \Rightarrow \frac{1}{2} M_U U^2 = \int (\text{constant}) \]

\[ \Rightarrow u = 2g, \] no the bubble accelerates (upward) at \( 2g \).

Two-dimensional Potential Flow

(i) Point Source

\[ \nabla \cdot \Phi = \frac{1}{2\pi} \left( r \frac{\partial \Phi}{\partial r} \right) = 0 \] for anti-symmetric flow.

\[ \Rightarrow \frac{1}{2\pi} \frac{\partial \Phi}{\partial r} = \frac{\pi}{2\pi} \Phi = g \] - constant source strength.

\[ \Rightarrow \Phi = \frac{\pi}{2\pi} g r \] - separate stream function.

(ii) General Solution in Plane Polar coordinates

\[ \Phi = \frac{1}{2\pi} \ln r + \frac{1}{2} \Phi + \frac{1}{2} \left( A_0 r^2 + B_1 r - \cdots \right) \]

(iii) Point Vortex

\[ \Phi = \frac{1}{2\pi} \ln r \]

Exercise: Show that \( \nabla \times \Phi = 0 \) if \( r \neq 0 \)

and that \( \oint \Phi \cdot dl = \left\{ \begin{array}{ll} 0 & \text{if the origin is inside } C \\ \text{otherwise} & \end{array} \right. \)

The circulation \( \oint \Phi \cdot dl = \int \omega \cdot ds \) is the measure of the vorticity enclosed within \( C \).
(ii) Uniform Flow past a circle (or cylinder)

\[ \nabla^2 \psi = 0 \quad (r > a) \]

\[ \psi = U(1 + \frac{a}{r}) \cos \theta + \frac{k}{2\pi} \theta \]

No net source of fluid, so \( \phi_r = 0 \) but we must allow a non-zero \( k \) to account for any vorticity in the viscous boundary layer near the surface of the circle.

Velocity \( u_\theta = \frac{\partial \phi}{\partial r} = U (1 - \frac{a^2}{r^2}) \cos \theta \), \( u_\phi = \frac{\partial \phi}{\partial \phi} = -U \left( 1 + \frac{a^2}{r^2} \right) \sin \theta + \frac{k}{2\pi} r \)

Streamfunction (Cf Sheet 1, 28)

\[ \psi = u_\theta \sin \theta (1 - \frac{a^2}{r^2}) - \frac{k}{2\pi} \ln \left( \frac{r}{a} \right) \]

\[ k = 0 \]

\[ 0 < k < 4\pi \text{ca} \]

\[ 4\pi \text{ca} < k \]

Note: For steady potential flow, Bernoulli's works everywhere, not just along streamlines.

So \( \rho \omega + \frac{1}{2} \rho U^2 = \rho + \frac{1}{2} \rho \left( \frac{k^2}{2\pi a} - 2U \sin \theta \right)^2 \) on the surface of the circle.

\[ \rho = \rho_0 + \frac{1}{2} \rho U^2 - \frac{p k^2}{8\pi a} + \frac{p k}{\pi a} \sin \theta + \frac{p k}{\pi a} \sin^2 \theta - 2pU \sin \theta \]

Symmetric fore and aft, so that there is no net force in the flow direction.
Force in the $y$ direction:
\[ F_y = - \int_0^{2\pi} \rho \sin \theta \ a \ d\theta = - \int_0^{2\pi} \pi \rho \sin^2 \theta \ \sin \theta \ \ d\theta = -\rho \text{U} K \]

In general, in "magnetic force" resulting from the interaction between a flow $U$ and a vortex $K$ is $E = \rho \text{U} \times K$.

Aerfoils generate circulation around the wing by having a sharp trailing edge, controlling separation.

Kutta Condition - the circulation is just sufficient to cause separation at the leading edge.

Pressure Field in (time dependent) potential flows

Euler Equation:
\[ \rho \left( \frac{\partial u}{\partial t} + \nabla \left( \frac{1}{2} \rho u^2 \right) - u \times \omega \right) = -\nabla p + \nabla \cdot \mathbf{F} \]

If $u = \nabla \Phi$, $\omega = \nabla \times \mathbf{v}$, then $\mathbf{F} = \nabla \Phi$.

and we have
\[ \nabla \left( \frac{\partial \Phi}{\partial t} + \frac{1}{2} \rho \left( \frac{\partial u}{\partial t} \right)^2 + p + X \right) = 0 \]
\[ \frac{\partial \Phi}{\partial t} + \frac{1}{2} \rho \left( \frac{\partial u}{\partial t} \right)^2 + p + X = f(t), \] independent of position.

In particular, if the flow is steady, then $H = \frac{1}{2} \rho \left( \frac{\partial u}{\partial t} \right)^2 + p + X$ is constant throughout the fluid domain.

Oscillations in a manometer

\[ u = \hat{a} = \hat{a} \]

Large cross section $\Rightarrow$ Slow flow $\Rightarrow$ $p \times$ constant $= 0$ \text{WLOG}

LHS $\phi \times u_y$: $\frac{\partial}{\partial t} = u_y = -\hat{y}$

RHS $\phi \times -u_y$: $\frac{\partial}{\partial t} = -u_y = -\hat{y}$

Consider pressure on the two free surfaces.

\[ \rho \hat{y} u_y + \frac{1}{2} \rho \left( \frac{\partial u}{\partial t} \right)^2 + p_s + \rho g h = -\rho \hat{y} u_y + \frac{1}{2} \rho \left( \frac{\partial u}{\partial t} \right)^2 + p_s - \rho g h \]

$\Rightarrow 2\rho \hat{H} \frac{\partial u}{\partial t} + 2\rho g h = 0$ $\Rightarrow \frac{\partial u}{\partial t} + \frac{\rho g h}{\hat{H}} = 0$
This is the equation for SHM with frequency $f = \frac{a}{2\pi}$.

Oscillations of a bubble

Consider the flow of fluid external to a bubble of radius $a(t)$. \[ \nabla^2 \psi = 0 \quad (r > a) \quad \psi \to 0 \quad (r \to \infty) \]
Then \[ \psi = \frac{A}{r} \quad \Rightarrow \quad -\frac{A}{r^2} = \ddot{a} \quad \Rightarrow \quad \psi = -\frac{a^2 \ddot{a}}{r} \]

Ignore gravity and compare a point on the surface of the bubble with the far-field.

Note\[ \frac{4}{2} \int_{\rho}^{a} \frac{\rho^2}{r^2} \psi(r) \, d\rho = \frac{a^2}{2} \frac{\ddot{a}}{a} \quad \text{Note:} \quad \int_{\rho}^{a} \frac{\rho^2}{r^2} \psi(r) \, d\rho = -\frac{a^2 \ddot{a}}{r} \]

\[ \Rightarrow \quad -\rho \left( a^2 \ddot{a} + 2 \dddot{a}^2 \right) + \frac{1}{2} \rho \ddot{a}^2 + \rho = \rho_0 \quad (t) \]

Small Oscillations of gas bubbles.

\[ a = a_0 + \eta(t) \quad \eta < a_0 \]
\[ \rho \left( a^2 + \frac{r}{r} \right) = \rho \left( a_0 + \eta(t) \right) \quad \text{or} \quad \rho \ddot{a}^2 = -\frac{\rho_0}{a_0} \]

If oscillations are adiabatic, then \[ \frac{PV^n}{R} = \text{constant} \quad \text{where} \quad n = \frac{\gamma - 1}{\gamma - 1/2} \quad \text{is the ratio of specific heats} \]

Therefore\[ \rho \ddot{a}^2 = -\frac{\rho_0}{a_0} \quad \Rightarrow \quad \ddot{a}^2 = -\frac{\rho_0}{a^{n-1}} \]

\[ \Rightarrow \quad \eta + \frac{\rho_0}{a^{n-1}} \eta = 0 \]

SHM with frequency\[ \omega = \sqrt{\frac{\rho_0}{a^{n-1}}} \]

Frequency \[ 2 \times 10^{-2} \quad \text{for a} \quad 1 \text{cm}^3 \text{bubble in water.} \]

High frequency, no the adiabatic assumption is valid.
Fluid Dynamics

Water Waves

\[ \mathbf{z} = 1(x, y, t) \]

Assume that the water is inviscid, and motion is started from rest. Then the flow is irrotational, \( \mathbf{a} = \nabla \phi \), and \( \nabla^2 \phi = 0 \) - Hooke's Law

1. Kinematic Boundary Conditions:
   - At the rigid base: \( \mathbf{w} = \frac{\partial \mathbf{z}}{\partial z} = 0 \) (\( z = -H \))
   - At the free surface:
     \[ \begin{align*}
     \frac{\partial \mathbf{z}}{\partial t} + u \frac{\partial \mathbf{z}}{\partial x} + v \frac{\partial \mathbf{z}}{\partial y} &= \mathbf{w} \quad (z = h) \\
     \frac{\partial \mathbf{z}}{\partial x} + \frac{\partial \mathbf{z}}{\partial y} &= \frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \\
     \frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} &= \frac{\partial \mathbf{z}}{\partial t} + \frac{\partial \mathbf{z}}{\partial x} + \frac{\partial \mathbf{z}}{\partial y} \end{align*} \]

2. Dynamic Boundary Conditions:
   - \( p = p_0 \) (\( z = h \))

3. Use expression for pressure in time-dependent potential flow:
   \[ \frac{\partial^2 \mathbf{z}}{\partial t^2} + \frac{1}{2} \nabla^2 (\mathbf{z}^2) + \mathbf{p}_0 + \rho g h = f(t) \] (\( z = h \))

4. Linearization: Assume \( h \ll H \) \( \frac{\partial \mathbf{z}}{\partial x} \ll 1 \)

5. Also assume that \( h \ll 1 \) is small. In consequence of these conditions:
   - Ignore terms of quadratic order or higher in disturbance quantities.
   - Use Taylor expansions to write (for example)
     \[ \begin{align*}
     \frac{\partial^2 \phi}{\partial x^2} \mathbf{z} & = \frac{\partial^2 \phi}{\partial x^2} \mathbf{z} = 0 + \frac{\partial^2 \phi}{\partial x^2} \mathbf{z} \mathbf{z} + \frac{\partial^2 \phi}{\partial x^2} \mathbf{z} \mathbf{z} \mathbf{z} + \cdots \\
     \end{align*} \]

6. Linear Water Waves
   \[ \nabla^2 \phi = 0 \quad (-H < z < 0) \]
   \[ \frac{\partial \phi}{\partial z} = 0 \quad (z = -H) \]
   \[ \frac{\partial \phi}{\partial z} = \frac{\partial \phi}{\partial z} \quad (z = 0) \]
   \[ \rho \frac{\partial \mathbf{z}}{\partial t} + \mathbf{p}_0 + \rho g h = f(t) \quad (z = 0) \]

7. Look for 2D solutions, (30 is not much harder) in the form of a single Fourier mode (traveling sine wave):
   \[ h = h(x, t) = h_0 e^{i(kx - \omega t)} \]
   \[ (\omega \text{ here is the angular frequency, and } k \text{ is the horizontal wave number}) \]

8. \[ \phi(x, z, t) = \phi(z) e^{i(kx - \omega t)} \]
Substitute into Laplace's equation:

\[ k^2 \psi + \psi'' = 0 \quad \psi' = 0 \text{ at } z = -H \]

\[ \Rightarrow \psi = \psi_0 \cosh k(z + H) \]

Boundary conditions at \( z = 0 \) give

\[ -i \omega \psi_0 = k \psi_0 \sinh kH \]

\[ -i \omega \psi_0 \cosh kH + g \psi_0 = 0 \] (4,5)

Combining the last two equations:

(Note: these are homogeneous eigenvalue equations for \( \omega(k) \). They do not determine amplitudes \( \psi_0, \psi_0 \))

\[ -i \omega \psi_0 \cosh kH + g \k \psi_0 \sinh kH = 0 \]

\[ \Rightarrow \frac{\omega^2}{k} = g k \tanh kH \]

Dispersion relation

Note that waves of different wavenumbers \( k \) have different frequencies \( \omega \).

Phase Speed - the speed at which wave travels, for example

\[ c = \frac{\omega}{k} \]

\[ e^{ik(x-ct)} = e^{ikx - i\omega t} \]

So

\[ \omega^2 = \frac{k^2}{c^2} \tanh kH \]

and \( c \) depends on wavelength.

Short waves (deep water)

wavelength \( \lambda = \frac{2 \pi}{k} \Rightarrow \frac{kH \gg 1}{\tanh kH \approx 1} \Rightarrow \omega \approx \frac{\sqrt{gH}}{k}, \ c \approx \sqrt{\frac{gH}{k}} \]

Long waves (shallow water)

\( \lambda \gg H, \ kH \ll 1, \ tanh kH \approx kH \)

\[ \Rightarrow \omega \approx \sqrt{gH}, \ c \approx \sqrt{gH}, \ \text{constant} \]
Fluid Dynamics

Water Waves
\[ c^2 = gk \tan kh \]

**Note:**
- Unlike light and sound, water waves are dispersive; waves of different length travel at different speeds.

Distant Storm

Smooth waves arrive at the beach, long waves first.

- Shallow-water waves (long waves) are approximately non-dispersive
- e.g., Tsunamis: \( H \approx 4 \text{ km}, \ A \approx 500 \text{ km} \), shallow water waves

The constant wave speed \( c = \sqrt{gH} \approx \frac{10 \times 4 \times 10^2}{500^2} = 200 \text{ m/s} \)

This is roughly the speed of a commercial airplane.

The group velocity (velocity of energy propagation) is \( G = \frac{dc}{dk} \)

(See Part II Waves, or Asymptotic methods for proof)

So for deep water waves

\[ c = \sqrt{g \frac{1}{\sqrt{\frac{2}{k}}} \quad G = \frac{1}{\sqrt{\frac{2}{k}}} = \frac{c}{k} \]

The difference between the phase speed and the group velocity gives rise to the pattern of ship waves. For example:

**Exercise:** Re-work the analysis specifically for deep water waves using the boundary condition \( \psi = \text{constant} \) at \( z = \pm \infty \)

(Study, \( \psi \rightarrow \text{constant} \) at \( z = \pm \infty \) )

**Boundary Waves** e.g., in a square-cylinder

Linearized equations:

\[ \begin{align*}
\frac{\partial^2 \psi}{\partial z^2} + \frac{\partial^2 \psi}{\partial x^2} &= 0 \\
\frac{\partial \psi}{\partial z} &= 0 \\
\frac{\partial \psi}{\partial x} &= 0 \\
\psi &= 0 \text{ at } x = 0, z = 0 \text{ and at } z = \pm \infty
\end{align*} \]
Separation of Variables:

\[ y = y_0 e^{\frac{\hbar}{a} \left( \cos \left( \frac{\hbar}{a} \right) - \cos \left( \frac{\hbar}{a} \right) \right) + \frac{\hbar}{a} \left( \cos \left( \frac{\hbar}{a} \right) - \cos \left( \frac{\hbar}{a} \right) \right)} \]

The only difference is that \( \hbar \) is quantized by the side wall. These are standing waves.

Rayleigh-Taylor Instability

The solution (in water) is exactly as for water waves but with \( \omega = \sqrt{\gamma} \).

Therefore, \( \omega^2 = -g/h \), \( \omega = i \cdot \sqrt{\gamma} \) \( \Rightarrow h = i \cdot \sqrt{\gamma} e^{\pm \frac{\gamma}{a}} \).

\( \omega \) grows exponentially and \( h \) decays exponentially.

Random Perturbation (initial condition) will have \( A \neq 0 \), so the disturbance will grow in time.

Rayleigh-Taylor Instability with surface tension

Consider water waves first, \( \rho = \rho_0 + 2 \gamma r \) curvature

\( \rho = \rho_0 + \gamma \cdot \text{curvature} \cdot r \)

\( \rho_0 - r \cdot \gamma \)

surface tension

If \( \frac{\gamma}{r} < 1 \), so the deformed dynamic boundary condition becomes

\( \frac{\rho_0}{r} + \gamma \cdot \frac{\partial^2}{\partial x^2} = \frac{\gamma}{r} \) \( \Rightarrow = 0 \)

In deep water:

\( \omega^2 = \gamma \cdot k + g \)

This is the dispersion relation for capillary-gravity waves.

For the Rayleigh-Taylor instability:

\( \gamma^2 \Rightarrow k^2 - \frac{1}{\gamma^2} \)

Instability if \( \omega^2 < 0 \Rightarrow k^2 < \frac{1}{\gamma^2} \)

\( \Rightarrow (n^2 + m^2) r^2 < \frac{\gamma^2}{\rho} \)

Not unstable if \( n = 1, m = 0 \) (say)

\( \Rightarrow a^2 > \frac{\rho}{\gamma} \)

\( \Rightarrow a > \frac{\rho}{\gamma} \)

\( \Rightarrow 9 \text{ km for water over air} \)
Fluid Dynamics in a Rotating Frame

The Lagrangian (particle) acceleration in a rotating frame of reference is
\[ \frac{\partial \mathbf{a}}{\partial t} + 2 \frac{\partial \mathbf{a}}{\partial x} + \mathbf{a} \times (\Omega \times \mathbf{x}) \]  
(See Dynamics)

So \[ \rho \left( \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} + \mathbf{a} \times (\Omega \times \mathbf{x}) \right) = -\nabla P - \rho \mathbf{a} \times (\Omega \times \mathbf{x}) + \rho \mathbf{g} \]

where \( \mathbf{g} \) is the rotation rate of the frame.

Note that \( \rho \mathbf{a} \times (\Omega \times \mathbf{x}) \) is a constant (in time) and conservative body force.

So the centrifugal force just adds to the hydrostatic pressure, so globally, the sea surface, for example, is not spherical.

For Earth, \( \mathbf{g} = \frac{2\pi}{\hbar^2} \mathbf{S} \), largest scale \( \mathbf{L} \approx 10^4 \text{ km} \)

So \[ \frac{\mathbf{g}}{2\pi x (\Omega \times \mathbf{x})} \approx 10^{-4} \mathbf{S} \]

So we will ignore the centrifugal term.

Consider motions for which \( \mathbf{u} \cdot \nabla \mathbf{u} \ll \mathbf{u} \times \mathbf{g} \)

\[ \Rightarrow \mathbf{u} \cdot \mathbf{g} \ll \mathbf{u} \times \mathbf{g} \]

Relative vorticity \( \ll \) planetary vorticity.

E.g., an atmospheric weather system, \( \mathbf{u} = 10 \text{ m s}^{-1}, \mathbf{L} \approx 10^3 \text{ km} \)

\[ \Rightarrow \mathbf{u} \cdot \mathbf{g} = \frac{10^6}{2\pi \times 10^5} \times \frac{1}{2\pi} \ll \mathbf{u} \times \mathbf{g} \]

It is a reasonable approximation to ignore the advection term.

With these two approximations, \( \partial \mathbf{u} + 2 \frac{\partial \mathbf{u}}{\partial x} + \mathbf{u} \times (\Omega \times \mathbf{x}) = -\nabla P + \mathbf{g} \)

This is the Euler equation at small Rossby number \( R_o = \frac{\mathbf{u}}{2\pi x \times \mathbf{g}} \). This approximation relates to

- strong rotation
- low fluid speeds
- large length scales

It is conventional to write \( 2\Omega = \mathbf{f} \), \( \mathbf{f} \) called the "planetary vorticity" or the "Coriolis Parameter". It is also conventional in this subject to use \( \mathbf{S} \) for the relative vorticity \( \nabla \times \mathbf{u} \).
Shallow Water Equations

Consider a layer of fluid of depth \( h(x, y) \) with \( \rho = \rho_0 \) on \( z = h(x, y) \). The surface of the ocean or a constant pressure surface defining the top of the atmosphere.

\[ \nabla \cdot \mathbf{u} = 0 \implies \frac{\partial w}{\partial z} = -\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \quad h = Z \]

\[ w \ll u, v \]

Consider \( \mathbf{u} = (u, v, 0) \), \( F = (0, 0, F) \)

\[ \rho \frac{\partial u}{\partial t} + \rho f v = -\frac{\partial p}{\partial x} \]

\[ \rho \frac{\partial v}{\partial t} + \rho f u = -\frac{\partial p}{\partial y} \]

\[ 0 = -\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} - \rho g \]

From \( \mathbf{3} \), \( \rho = \rho_0 + \rho g \left[ h(x, y) - Z \right] \)

Horizontal momentum equations

1. \( \rho \frac{\partial u}{\partial t} - \rho f v = -\frac{\partial p}{\partial x} \)
2. \( \rho \frac{\partial v}{\partial t} + \rho f u = -\frac{\partial p}{\partial y} \)

Note that with our shallow water approximation, pressure is hydrostatic.

- Note also that horizontal accelerations are independent of \( z \).
- Initial conditions are usually such that \( u, v \) are also independent of \( z \).
- Horizontal pressure gradients are proportional to horizontal variations in \( h \).

Geostrophic Balance

In steady state, \( u = \frac{1}{\rho_0} \left( -\frac{\partial p}{\partial x} \right) = \frac{\partial \phi}{\partial x} \) where \( \phi \) is the pressure term.

\( v = \frac{1}{\rho_0} \left( -\frac{\partial p}{\partial y} \right) = -\frac{\partial \phi}{\partial y} \)

So the 2D streamfunction \( \phi = -\frac{\partial \psi}{\partial y} \) where \( \psi \) is the height.

Therefore, pressure (height) contours are streamlines.

"Cyclonic" winds

In the Northern hemisphere, with the wind on your back and our left on your right.

"Anticyclonic" winds
Fluid Dynamics

Tangential plane approximation

\[ \mathbf{f}(x, y) = (u, v, 0) \]

\[ x = (0, 0, f) \]

Mass Conservation

Consider a cylinder with horizontal cross-section \( D \)

\[ \int_0^1 \rho \mathbf{v} \cdot dS = -\int_D \frac{\partial p}{\partial t} dV \]

This is true for arbitrary domains, \( \mathbf{v} \cdot \nabla \rho \cdot (\mathbf{v} \cdot \nabla) = 0 \)

Note that we are still assuming \( \rho = \text{constant} \)

In Cartesian components, \( \frac{\partial^2 \mathbf{v}}{\partial x^2} + \frac{\partial^2 \mathbf{v}}{\partial y^2} = 0 \) . Note \( \mathbf{v} \cdot \nabla \mathbf{v} \equiv \mathbf{C} \)

Linearized equations of motion

Suppose \( h = h_0 + \eta(x, y, t) \) \( \eta \ll h \)

Then \( \frac{\partial^2 \eta}{\partial t^2} + \omega^2 (\frac{\partial^2 \eta}{\partial x^2} + \frac{\partial^2 \eta}{\partial y^2}) = 0 \) (dropping the subscript \( H \))

1. Exercise: For non-rotating, shallow water waves \( (f = 0) \)

show that \( \frac{\partial^2 \eta}{\partial t^2} - \rho_0 \left( \frac{\partial^2 \eta}{\partial x^2} + \frac{\partial^2 \eta}{\partial y^2} \right) = 0 \)

Wave equation with wave of speed \( c = \sqrt{\frac{g}{h_0}} \)

2. Eliminate \( \eta \) from (2) by taking the curl:

\[ \frac{\partial^2 \mathbf{v}}{\partial x^2} + (\nabla \times \mathbf{v}) = 0 \]

where \( S = (\frac{\partial^2 - \frac{\partial^2}{\partial y^2}}{2}) \) \( \mathbf{v} \) is the relative vorticity

Substitute for \( \nabla \times \mathbf{v} \) from (2)

\[ \Rightarrow \frac{\partial^2 \mathbf{v}}{\partial x^2} = 0 \]

3. Potential Vorticity \( Q = S - \frac{\rho_0}{h_0} F \) is constant in time at each location in space

\[ Q = Q_0 (x, y, t) \] from initial condition

Aside in general, not linearised or rapidly rotating

\[ \left( \frac{\partial}{\partial t} (S + F) \right) = 0 \]

The quantity \( S - \frac{\rho_0}{h_0} F \) is the linearised form of the total potential vorticity
Take the divergence of \( \mathbf{D} = \frac{1}{\alpha} \nabla (\nabla \cdot \mathbf{u}) - \mathbf{E} \cdot \mathbf{D} + \mathbf{S} \), where \( \mathbf{S} = \nabla \times \mathbf{D} \) is the relative vorticity.

Substitute for \( \nabla \cdot \mathbf{u} \) from (2):

\[
\frac{1}{\alpha} \nabla^2 \mathbf{D} = \frac{1}{\alpha} \nabla^2 \mathbf{D} - \mathbf{E} \cdot \mathbf{D} = \mathbf{S} = \nabla \times \mathbf{D} \text{ is the relative vorticity.}
\]

Use conservation of potential vorticity to write \( \mathbf{S} = \mathbf{Q}_0 + \frac{\eta_0}{\alpha} \mathbf{F} \)

\[
\Rightarrow \frac{1}{\alpha} \nabla^2 \eta + \mathbf{F} \cdot \nabla \eta = -\frac{1}{\alpha} \mathbf{E} \cdot \mathbf{Q}_0, \quad \text{independent of time.}
\]

Note that the unforced equation \((\mathbf{Q}_0 = 0)\) supports waves with \( \eta \mathbf{C} \in i \omega t \).

**Example.** Suppose that there is a region of high pressure next to a region of low pressure \( \rho_0 + \eta_0 \). \( \eta_0 \) is the non-rotating case, we get waves traveling in both directions.

In the rotating case, there is a non-trivial, geostrophically balanced steady flow. Note that

\[
\mathbf{Q}_0 = \frac{\eta_0}{\alpha} \mathbf{F} = F \frac{\eta_0}{\alpha} \mathbf{x} \quad \text{in } \alpha > 0
\]

\[
\frac{\partial \eta}{\partial t} - \frac{1}{\alpha} \nabla^2 \eta + \mathbf{F} \cdot \nabla \eta = F \frac{\eta_0}{\alpha} \mathbf{x}, \quad \text{in } \alpha > 0
\]

Steady (geostrophically balanced) flow has \( \eta = \eta(x, t) \)

\[
\eta'' - \frac{\alpha \eta}{\mathbf{F}} = \frac{\eta_0}{\alpha} \mathbf{F}
\]

where \( \mathbf{R} = \frac{\mathbf{F}}{\eta_0} \), is the Rossby radius (of deformation).

Solve with \( \eta > 0 \) as \( x \to \pm \infty \).

\[
\eta, \eta' \text{ continuous at } x = \infty
\]

\[
\mathbf{F} \text{ Continuity of } \eta
\]

This gives continuity of pressure.
\[ \eta = \begin{cases} \eta_0 \left(1 - e^{-x/k}\right) & x > 0 \\ -\eta_0 \left(1 - e^{x/k}\right) & x > 0 \end{cases} \]

\[ u = \frac{2}{\rho g} \left(-\frac{x}{E}\right) = 0 \quad \text{low p} \quad \frac{x}{E} \quad \text{high p} \]

The Rossby radius gives the characteristic length scale for balanced flow in the atmosphere and oceans.

In plan-view:

low p -- \| -- High p

This is like a section through

\[ \text{Diagram of section through flow} \]