Some Topics

1. Brouwer's Fixed Point Theorem and related results.

Theorem

If \( f : D \to D \) is a continuous map from the closed unit disc \( D \subseteq \mathbb{R}^2 \) to itself, then \( f \) has a fixed point \( f(x) = x \).

Related issues: winding numbers, another proof (without complex analysis) of the Fundamental Theorem of Algebra (links to Alg. Top)

2. Approximation by polynomials. Any continuous function \( f : [0,1] \to \mathbb{C} \) (Stone-Weierstrass) can be approximated as closely as necessary by polynomials.

Links to Linear Analysis, Introduction to interpolation and numerical integration (Chebyshev Polynomials etc)

3. Some number theory topics such as the irrationality of \( e \) and \( \pi \), the construction of transcendental numbers, Continued fractions and related matters.

4. Baire Category Theorem and weird counterexamples, e.g. continuous nowhere-differentiable functions.

Brouwer's Fixed Point Theorem

In the proof of this theorem, we will make crucial use of the Bolzano-Weierstrass property: If \( X \subseteq \mathbb{R}^2 \) is closed and bounded, and \( (x_i)_{i=1}^\infty \subseteq X \), then we may select a subsequence of the \( x_i \) converging to a point of \( X \).
The Bolzano–Weierstrass Property is most naturally discussed in the context of a general metric space, where it is known as sequential compactness, the same thing as compactness. Recall that a metric space is a set \( X \) equipped with a distance function \( d : X \times X \to \mathbb{R}_{\geq 0} \), such that

i) \( d(x, y) = 0 \iff x = y \)

ii) \( d(x, y) = d(y, x) \)

iii) \( d(x, z) \leq d(x, y) + d(y, z) \)

The crucial example here is \( X \subseteq \mathbb{R}^n \), with \( d \) the Euclidean metric.

**Cauchy Sequences**

Let \( (x_n)_{n=1}^{\infty} \subseteq X \) be a sequence of elements of \( X \).

Formally, for all \( E > 0, \exists N \) such that if \( n, m \geq N \), then \( d(x_n, x_m) > E \). "A sequence that is trying to converge."

Cauchy sequences need not converge, e.g. \( X = (0, 1], \)
\( x_n = \frac{1}{n} \), and also \( X = \mathbb{Q}, x_n = 3.1415926... \)

A metric space is said to be complete if all Cauchy sequences do converge. That is, if \( (x_n)_{n=1}^{\infty} \subseteq X \) is a Cauchy sequence, then there is a (unique) point \( x \) such that \( x_n \to x \).

Formally, this means that for all \( E > 0 \), there exists \( N \) such that if \( n \geq N \) then \( d(x_n, x) < E \).
The real numbers $\mathbb{R}$ can be constructed as the completion of $\mathbb{Q}$.

The real numbers $\mathbb{R}$ are the unique complete metric containing the rationals $\mathbb{Q}$ (with the usual metric) as a dense subset (every real number can be approximated by rationals). How do we construct $\mathbb{R}$?

Define $\mathbb{R}$ to be the set of Cauchy sequences in $\mathbb{Q}$, subject to an equivalence relation, namely that $(x_n)_{n=1}^{\infty} \sim (y_n)_{n=1}^{\infty}$ precisely if $x_n - y_n \to 0$. To really set up the real numbers $\mathbb{R}$, one must do many things. Define a distance on $\mathbb{R}$,

$$d((x_n)_{n=1}^{\infty}, (y_n)_{n=1}^{\infty}) = \lim_{n \to \infty} d(x_n, y_n),$$

and prove this is well defined. Then one must prove that $\mathbb{R}$ is complete, which gets notationally nasty. Other constructions are Dedekind cuts. However, the completion construction is much more general; one can take any metric space $X$ and form its completion $\bar{X}$ which is then a complete metric space and contains $X$ as a dense subset. A particular example is the $p$-adics $\mathbb{Q}_p$ ($p$ prime) with respect to the metric defined by $d(x, y) = \mid x - y \mid_p$ where

$$\mid t \mid_p = p^{-n} \text{ if } t = \frac{a}{b} \text{ with } (a, p) = (b, p) = (a, b) = 1.$$

Note, for example that $2^n \to 0$ in $\mathbb{Q}_2$.

**Definition**

A metric space $X$ is **sequentially compact** (has the Bolzano-Weierstrass property) if the following holds:
Every sequence in $X$, $(x_n)_{n=1}^{\infty}$, has a convergent subsequence $x_{n_1}, x_{n_2}, \ldots$, with $x_{n_i} \to x$.

**Theorem**

The following are equivalent for a metric space $X$:

1. $X$ is sequentially compact.

2. $X$ is complete and totally-bounded, which means that for every $\varepsilon > 0$, $X$ can be covered by finitely many balls of radius $\varepsilon$.

3. $X$ is compact; every open cover has a finite sub-cover.

**Very Sketchy Proof** (Sutherland's Metric and Topological Spaces)

$1 \Rightarrow 2$ is relatively easy. If there is a Cauchy sequence which does not converge, then it has no convergent subsequence. If it is not totally bounded, then there is some $\varepsilon > 0$ with an infinite sequence of points $x_n$, each pair $\geq \varepsilon$ apart. This sequence clearly has no convergent subsequence.
Sequential Compactness / Bolzano-Weierstrass Property

\[(X \text{ is sequentially compact}) \iff (X \text{ is complete and totally bounded})\]

\[\Rightarrow\, i) \text{ Suppose that } (x_n)_{n=1}^{\infty} \subseteq X. \text{ Cover } X \text{ by finitely many balls of radius } 1. \text{ One of these, } B_1, \text{ contains infinitely many of the } x_n. \]

\[\text{Cover this by finitely many balls of radius } \frac{1}{2}. \text{ One of these, } B_2, \text{ contains infinitely many } x_n. \text{ Continue with balls of radius } \frac{1}{4}, \frac{1}{8}, \ldots\]

Select one point of \((x_n)_{n=1}^{\infty}\) from each ball. This is a Cauchy sequence. Since \(X\) is complete, this subsequence converges. \(\square\)

Last time we talked about \(\mathbb{R}\), which was complete (essentially by definition). It follows quite easily that \(\mathbb{R}^n\) is complete (if you have a Cauchy sequence, look at it coordinate-wise; each coordinate converges and this gives the limit).

\[\text{Claim: } \text{seq. compact } \implies \text{has Bolzano-Weierstrass property}\]

\(\mathbb{R}^n\) is compact if and only if \(X\) is closed and bounded.

\[\text{Proof:}\]

In \(\mathbb{R}^n\), totally bounded is the same as bounded (divide into cubes of side length \(\varepsilon\)). (N.B. This uses the fact that \(\mathbb{R}^n\) is finite dimensional, and is not true in general. There is a counterexample on the example sheet.)

In a general complete metric space, a subset is complete if and only if it is closed. If you have a Cauchy sequence in the
subset, then it converges in the big space. But then the limit is in the subset. Conversely, if your subset is not closed, pick a point $x \in \text{CL}(X) = \mathbb{R}$, but $x \notin X$. Pick a sequence $(x_n)_{n=1}^{\infty} \subseteq X$ and $x_n \in B(x, \frac{1}{n})$. This is then a Cauchy sequence in $X$, which converges to a point not in $X$, namely $\infty$.

Brouwer's Fixed Point Theorem

Theorem

Suppose that $f : D^n \rightarrow D^n$ is continuous, where $D^n$ is the closed unit ball in $\mathbb{R}^n$. Then $f$ has a fixed point. This is equivalent to the following

Proposition

There does not exist a continuous "retraction" $g : D^n \rightarrow \partial D^n = S^{n-1}$. That is to say $g(x) = x$ if $x \in \partial D^n$.

Proposition $\Rightarrow$ Theorem

Suppose that we have a map $f : D^n \rightarrow D^n$ without fixed points.

Then we define a map $g : D^n \rightarrow \partial D^n$ by taking the directed line segment from $f(x)$ to $x$ and extending it to meet $\partial D^n$ (see picture). This map is continuous (example sheet) and preserves the boundary. Hence it is a retraction $X$

Theorem $\Rightarrow$ Proposition

If you have a continuous retraction $g : D^n \rightarrow \partial D^n$, compose it with a continuous map from $S^{n-1}$ to $S^{n-1}$ without fixed points.

(When $n=2$, any non-trivial rotation is fixed as is $x \mapsto -x$ for any $n$.)
Equivalent version of Proposition

There does not exist a continuous map $f : T^n \to \partial T^n$ which fixes the boundary. Here, $T^n$ is the $n$-simplex, which when $n=2$ is just an equilateral triangle.

Proof of Equivalence

Suppose that we have a continuous retraction $f : T^n \to \partial T^n$. We can use this to make a continuous retraction of the ball.

A triangulation of the simplex $T^n$ is something like this:

A division of the $n$-simplex (when $n=2$, the triangle) into sub-n-simplices (sub-triangles) which if they meet, do so in a face (when $n=2$, the edges, vertices).

A Sperner-colouring of a triangulation is an assignment of colours \(\{0, 1, \ldots, n\}\) to the vertices of the triangulation such that

i) The $n+1$ vertices of $T^n$ get the colours $0, 1, \ldots, n$ once each.
ii) If $v \in \partial T^n$, then $v$ must have the same colour as one of the vertices of the $n-1$ dimensional face in which it lies.
iii) If $v$ lies in the interior, its colour can be arbitrary.
Sperner's Lemma

If you Sperner-colour a triangulation of $T^n$ then there is an elementary sub-simplex whose vertices are labelled $0, 1, \ldots, n$ with not further each colour appearing exactly once.

Odd number of such simplices
Proof of Sperner's Lemma

The base case where \( n = 1 \) is fairly obvious.

The multi-colored sub-simplices are just the segments with different colored end-points.

Suppose now that \( n \geq 2 \). If \( S \) is an elementary sub-simplex in our triangulation, define \( F(S) \) to be the number of \((n-1)\)-faces of \( S \), colored with all of \( \{0, \ldots, n-1\} \). For example, in the picture, \( F(S_1) = 0 \), \( F(S_2) = 2 \).

Note that \( F(S) \equiv 0 \) (2) unless \( S \) is multicolored.

So the number of multicolored sub-simplices is \( \sum_S F(S) \) (2).

We can evaluate \( \sum_S F(S) \) by looking at the \((n-1)\)-faces colored \( \{0, 1, \ldots, n-1\} \). Any internal \((n-1)\)-face is counted twice, so it makes no contribution mod 2. So the only \((n-1)\)-faces that count are on the boundary.

Hence, the number of multicolored \( n \)-simplices is congruent to

\[
\sum_S F(S) = 2^n \quad \equiv \quad \# \text{ boundary } (n-1) \text{-faces colored } \{0, 1, \ldots, n-1\}
\]

\( \equiv 1 \) by the inductive hypothesis, since the coloring of the \( \{0, 1, \ldots, n-1\} \) boundary face is also a Sperner coloring.

Back to the Brouwer Fixed Point Theorem:

Suppose we have a continuous map \( f : T^n \to T^n \) which fixes

the boundary. Triangulate the \( n \)-simplex into elementary sub-simplices
each with diameter \( < E \) (given some \( E > 0 \)).

We Spencer - Colour this triangulation. Assign labels \([0, 1, \ldots, n]\)

arbitrarily to the outside vertices. Let \( F_i \) be the \((n-1)\)-face

opposite the vertex \( i \). Given a vertex \( v \in \) the triangulation, colour it

with some colour \( i \) such that \( f(v) \notin F_i \). This is always possible

since \( F_1 \cap \ldots \cap F_n = \emptyset \). This is easily seen to be a Spencer-colouring

(If fixes vertices on the boundary, so we only have the correct colours to

choose from.)

By Spencer's Lemma, there is a multi-coloured sub-simplex. Suppose the

it has vertices \( X_0^{(e)}, \ldots, X_n^{(e)} \). Then the distance between each pair of

these points is \( < E \), and \( f(X_i^{(e)}) \notin F_i \), since \( X_i^{(e)} \) has

colour \( i \).

Take a sequence of values of \( E \) tending to 0, e.g. \( 1, \frac{1}{2}, \frac{1}{4}, \ldots \)

Look at the sequence \( X_0^{(e)}, X_0^{(e)}, \ldots, X_0^{(e)} \), all of which lie

in the closed bounded set \( \Gamma \). By Bolzano-Weierstrass, there is a

convergent subsequence \( X_0^{(e)}, X_0^{(e)}, \ldots \rightarrow \chi \).

Since \( E_j \rightarrow 0 \), we also have \( X_1^{(e)}, X_1^{(e)}, \ldots \rightarrow \chi \), and similarly

for \( X_2^{(e)}, \ldots, X_n^{(e)} \). Now, we coloured so that \( f(X_0^{(e)}) \notin F_0 \).

Hence \( f(X_0^{(e)}) \in F_1 \cup \ldots \cup F_n \). By continuity of \( f \) and closedness of

\( F_1 \cup \ldots \cup F_n \), we have \( f(\chi) \in F_1 \cup \ldots \cup F_n \). \( f(\chi) \in F_i, \) some \( i \).

But \( f(X_i^{(e)}) \rightarrow f(\chi) \).
Winding Number and Degree

Recall that if \( f: X \to Y \) is a map between metric spaces, then\( f \) is continuous at a point \( x \), if for all \( \varepsilon > 0 \), there is a \( \delta > 0 \) such that \( d(x, y) < \delta \Rightarrow d(f(x), f(y)) < \varepsilon \).

We say that \( f \) is uniformly continuous if the same \( \delta \) works for every \( x \). For example:

\[
f: \mathbb{R} \to \mathbb{R}, \quad x \mapsto x^2
\]

is not uniformly continuous.
Sperner’s lemma and Brouwer’s fixed point theorem

The purpose of this note is to correct the oversight made in lectures in connection with this deduction. Best to draw coloured triangles when trying to work through this!

Recall that Sperner’s lemma is the following.

**Theorem 1.** Suppose you triangulate the n-simplex $T^n$. Colour the vertices of the triangulation with colours 0, 1, ..., $n$ so that the $n + 1$ extremal vertices get each colour once, a vertex in an $(n - 1)$-face gets the same colour as one of its $n$ extremal vertices, and all other vertices are coloured arbitrarily (this is called a Sperner colouring). Then there is an elementary subsimplex (minimal simplex in the triangulation) which is multicoloured.

Recall also that we reduced Brouwer to the following statement:

**Theorem 2.** There is no continuous retraction $f : T^n \to \partial T^n$, that is to say a continuous map which preserves the boundary.

Let’s show how this follows from Sperner’s lemma (you are only really supposed to worry about the case $n = 2$).

Let $\varepsilon > 0$ be arbitrary. Then $T^n$ may be triangulated in such a way that each elementary subsimplex has diameter $< \varepsilon$. This is obvious by picture when $n = 2$, and the case $n \geq 3$ is on the example sheet. We’re going to make a Sperner colouring of this triangulation. First of all, colour the $n + 1$ extreme vertices of $T^n$ arbitrarily with each colour 0, 1, ..., $n$ being used once.

Now (and here we depart from lectures) I claim that there are closed sets $V_0, \ldots, V_n \subseteq \partial T^n$ with the following properties:

1. $V_0 \cup V_1 \cup \cdots \cup V_n = \partial T^n$;
2. If $F_i$ is the $(n - 1)$-face opposite the extreme vertex of $T^n$ which is coloured $i$, then $F_i \cap V_i = 0$.

In lectures I tried to take $V_i$ to be the union of all faces other than $F_i$, but this won’t work as (ii) is violated. I’ll leave you to convince yourself that such $V_i$ exist. When $n = 2$, so that $T^2$ is just a triangle labelled with 0, 1, 2, you can take $V_i$ to be the vertex $i$ together with the 90% nearest $i$ of the two edges containing $i$ (but not 100% as I tried to do in lectures!).

Now colour the vertices of the triangulation in such a way that if $v$ is coloured with $i$ then $v \not\in V_i$. By property (i) this is possible. Also, if $v$ lies in the face $F_i$ opposite $i$, then by (ii) we see that $v$ is not coloured with $i$. So this is a Sperner colouring.

Applying Sperner’s lemma, there is a simplex with vertices $x_0^{(e)}, \ldots, x_n^{(e)}$, with $x_i^{(e)}$ coloured $i$. 

1
Take a sequence of $\varepsilon$'s tending to 0, say $1, \frac{1}{2}, \frac{1}{4}, \ldots$. By Bolzano-Weierstrass there is a subsequence $x_0^{(\varepsilon_1)}, x_0^{(\varepsilon_2)}, \ldots$ tending to some point $x$. Each sequence $x_j^{(\varepsilon_1)}, x_j^{(\varepsilon_2)}, \ldots$ then tends to $x$ automatically.

Now $x_0^{(\varepsilon_i)}$ is coloured 0, and so $f(x_0^{(\varepsilon_i)}) \in V_0$. Since $V_0$ is closed and $f$ is continuous, we may take limits as $i \to \infty$ to conclude that $f(x) \in V_0$. Similarly $f(x) \in V_1, \ldots, V_n$. But $f(x)$ lies on the boundary $\partial T^n$, and so it must certainly lie in some face $F_i$ and hence not in $V_i$ by property (2).

BJG
Winding Numbers

1. Suppose that \( X \) is a (sequentially) compact metric space and that \( f: X \to Y \) is a continuous function. Then \( f \) is uniformly continuous.

Sketch Proof

If not, \( \exists \varepsilon > 0 \) such that \( \forall \delta > 0 \) works. In particular, for each \( n \) there are \( x_n, x'_n \) such that \( d(x_n, x'_n) \leq \delta \) but \( d(f(x_n), f(x'_n)) > \varepsilon \).

Since \( X \) is sequentially compact, there is a subsequence \( x_{n_k} \to x \).

But then \( x_{n_k} \to x \) as well. Hence by continuity, \( f(x_{n_k}) \to f(x) \).

2. With the same hypothesis, the image of a (sequentially) compact metric space under a continuous map is also (sequentially) compact.

Sketch Proof

Consider a sequence \( (f(x_n))_{n=1}^{\infty} \) in \( f(X) \). Then \( (x_n)_{n=1}^{\infty} \subset X \) has a convergent subsequence \( x_{n_k} \to x \). But then \( f(x_{n_k}) \to f(x) \) by continuity, thus \( (f(x_n))_{n=1}^{\infty} \) has a convergent subsequence.

We will be thinking about paths in \( \mathbb{R}^2 \), which we will frequently identify with \( \mathbb{C} \). Given \( z \in \mathbb{C} \setminus \{0\} \), a real number \( \Theta \) is called an argument of \( z \) if \( z = |z|e^{i\Theta} \). Note that \( \Theta \) is not uniquely defined; the allowable choices of \( \Theta \) differ by multiples of \( 2\pi \).

Lemma

Consider any domain in \( \mathbb{C} \) of the form \( X_a = \{ z = re^{i\Theta} | a-r < \Re \theta < a+r \} \). Then there is a continuous choice of argument on \( X_a \).
We call this arga. Given $Z$, define arga$(Z)$ to be the unique choice of argument in the range $(a-\pi, a+\pi)$.

**Brief Justification** that arga is continuous:

Suppose that $Z = re^{i\theta}$, $Z' = se^{i\phi}$. We need to show that if $Z, Z'$ are closed, then so are $\theta, \phi$. We have

$$|Z - Z'|^2 = |re^{i\theta} - se^{i\phi}|^2 = (r-s)^2 + 2rs(1-\cos(\theta - \phi))$$

Hence if $Z \approx Z'$, then $\cos(\theta - \phi) \approx 1$, but since $a-\pi < \theta, \phi < a+\pi$ and this implies that $\theta \approx \phi$.

**Corollary**

Suppose that $f: [0,1] \to X < C \setminus \{0\}$ is a path. Then there is a continuous choice of argument. Indeed, consider arga$f$. This is continuous since it is a composition of continuous functions.

**Proposition**

Let $f: [0,1] \to C \setminus \{0\}$ be a continuous path. Then there is a continuous choice of arg $(f(t))$.

**Proof**

The idea is to split into subpaths, each covered by the Corollary. Since $[0,1]$ is (sequentially) compact, so is $f([0,1])$. Hence, in particular, $f([0,1])$ is closed. Since $0 \notin f([0,1])$, there is some $\delta > 0$ such that $f([0,1])$ avoids $B_{\delta}(0)$, i.e., $|f(t)| > \delta$ for all $t \in [0,1]$.

By uniform continuity of $f$, we can split $[0,1]$ into sub-intervals $0 = t_0 < t_1 < \ldots < t_k = 1$. 

$\delta = \delta_0 < \delta_1 < \ldots < \delta_k = 1$
with $t_{i+1} - t_i \leq \delta$, where $\delta$ is chosen such that $|x - x'| \leq \delta$
\[ \Rightarrow |f(x) - f(x')| \leq \frac{\delta}{2}. \]

Looking at the picture, for $t$ in the range $[t_i, t_{i+1}]$, $f(t)$ always lies in the domain $X_\alpha$, where $\alpha$ is some choice of arg $f(t_i)$.

Hence there is a continuous choice of argument arg $f(t_i)$ for $t \in [t_i, t_{i+1}]$. Splicing these together, adjusting by multiples of $2\pi$ to make the endpoints match, we get a continuous choice of arg $f(t_i)$.

Suppose we now have two continuous choices of arg $f(t_i)$, $\Theta_1(t)$ and $\Theta_2(t)$. Then $\Theta_1(t) - \Theta_2(t) \in 2\pi \mathbb{Z}$ and this function is continuous. By the intermediate value theorem, $\Theta_1(t) - \Theta_2(t)$ is constant.

So arg $f(t)$ is unique up to a constant multiple of $2\pi$.

**Corollary**

If $f : [0, 1] \to \mathbb{C} \setminus \{0\}$ is continuous, then $\frac{1}{2\pi} \arg(f(1)) - \arg(f(0))$ does not depend on the choice of arg. If $f$ is a closed path, that is $f(0) = f(1)$, this quantity is an integer. We write this $w(f, 0)$ and call it the winding number or degree of $f$ about $0$.

**Observation**

$w(fg, 0) = w(f, 0) + w(g, 0)$. Indeed, if $f(t) = |f(t)| e^{i \arg(f(t))}$ and similarly for $g$, then...
\[ f(t)g(t) = |f(t)g(t)| e^{i(\arg(f(t)) + \arg(g(t)))} \]

is a continuous choice of \( \arg(fg) \). (Here, \( f, g : [0, 1] \rightarrow \mathbb{C} \setminus \{0\} \))

**Lemma** (Dog Walking Lemma)

Suppose \( f : [0, 1] \rightarrow \mathbb{C} \setminus \{0\} \) is a continuous path with \( f(0) = f(1) \).

Let \( g : [0, 1] \rightarrow \mathbb{C} \) be another continuous path with \( g(0) = g(1) \).

Suppose that \( |g(t)| < |f(t)| \) for all \( t \).

Then \( w(f + g, 0) = w(f, 0) \).

(\( f \): path of walker about the lamp-post, \( g \): displacement of the dog from the walker. Assume: the leash is short so the dog cannot touch the lamp-post.)

**Proof**

Note that \( w(f + g, 0) = w(f, 0) + w(l + \frac{g}{f}, 0) \).

However, \( (1 + \frac{g}{f}) \) always lies in the domain \( |z - 1| < 1 \) and hence in \( X_0 \). Hence \( \arg(l + \frac{g}{f}) \) is a continuous choice of argument taking values in \( -\pi < \theta < \pi \).

Hence, since \( \arg(l + \frac{g}{f})(1) \) and \( \arg(l + \frac{g}{f})(0) \) differ by \( 2\pi \mathbb{Z} \), they must be equal. So \( w(l + \frac{g}{f}, 0) = 0 \).
Dog Walking Lemma

If \( f : [0,1] \to C \setminus \{0\} \) has \( f(0) = f(1) \), and \( \exists g : [0,1] \to C \) with \( g(0) = g(1) \), \( f, g \) continuous and \( |g(t)| < |f(t)| \), then
\[
W(f+g,0) = W(f,0) \quad \text{(Very Reminiscent of Rouche's Theorem)}
\]

Homotopy Invariance of Winding Number

Suppose we have two closed paths \( f_0, f_1 : [0,1] \to C \setminus \{0\} \). Then we say that \( f_0 \) and \( f_1 \) are homotopic, if there is a continuous map ("Continuous movements of a closed path preserve the winding number.")

\[
F : [0,1] \times [0,1] \to C \setminus \{0\} \quad \text{such that}
\]

i) \( F(0,t) = f_0(t) \quad \forall t \)

ii) \( F(1,t) = f_1(t) \quad \forall t \)

iii) \( F(s,0) = F(s,1) \)

Idea

\( F(s,t) \), for fixed \( s \), is a closed path, which we could write \( f_s(t) \). The \( f_s, s \in [0,1], \) are a continuous family of paths, interpolating between \( f_0 \) and \( f_1 \).

Theorem

Suppose that \( f_0, f_1 : [0,1] \to C \setminus \{0\} \) are closed paths which are homotopic. Then \( W(f_0,0) = W(f_1,0) \)

Proof

The basic idea is to take a suitable subdivision \( 0 = t_0 < t_1 < \ldots < t_k = 1 \)
and check that \( W(f_0, 0) = W(f_{k+1}, 0) = \ldots = W(f_{k+1}, 0) = W(f_{k+1}, 0) \).

Note that \([0, 1] \times [0, 1]\) is sequentially compact, and hence closed. Hence, there is some \( \varepsilon > 0 \) such that \( |F(s, t) - F(s', t')| < \varepsilon \) for all \( s, t \in [0, 1] \times [0, 1] \). But \( F \) is continuous, and hence uniformly continuous, and so \( \exists \delta > 0 \) such that if \( d((s, t), (s', t')) \leq \delta \) then \( |F(s, t) - F(s', t')| < \varepsilon \). Suppose now that \( |s - s'| \leq \delta \).

Then \( |f_s(t) - f_{s'}(t)| < \varepsilon \) (since \( f_s(t) = F(s, t) \)) and \( |f_s(t) - f_{s'}(t)| < \varepsilon \).

Applying the Dog-Walking Lemma, \( f = f_s \), \( g = f_s' - f_s \), we get \( W(f_s, 0) = W(f + g, 0) = W(f, 0) = W(f, 0) \).

Now take a subdivision \( 0 = s_0 < \ldots < s_k = 1 \) with \( |s_i - s_{i-1}| \leq \delta \).

Then \( W(f_0, 0) = \ldots = W(f_1, 0) \).

Proposition

Write \( D \) for the closed unit disc in \( \mathbb{R}^2 = \mathbb{C} \). Suppose that \( g : D \to \mathbb{C} \) is a continuous function such that \( g \neq 0 \) on \( \partial D \), and the winding number of the path \( t \mapsto g(e^{2\pi i t}) \) is not zero. Then there is some \( z \in D \) such that \( g(z) = 0 \).

Proofs

Suppose not. Consider the map \( F : [0, 1] \times [0, 1] \to C \{0\} \) defined by \( F(s, t) = g(s e^{2\pi i t}) \). This is fairly clearly continuous. It gives us a homotopy between \( F(1, t) = g(e^{2\pi i t}) \) and...
and the constant path \( t \mapsto g(0) \). This is a contradiction, since by assumption the first path has non-zero winding number, but the winding number of a constant map is 0. \( \square \)

* In the world of complex analysis, the winding number of \( g(e^{2\pi i t}) \) is actually the number of zeroes of \( g \) inside \( D \) (counted with multiplicity).

* In Real Analysis this is not true.

**Another Proof of Brouwer’s Fixed Point Theorem in \( \mathbb{R}^2 \)**

It suffices to show that there is no continuous retraction from \( g : D \to \partial D \), that is to say that there is no continuous map which fixes \( \partial D \). However, such a map \( g \) has no zeroes in \( D \) and \( g(e^{2\pi i t}) = e^{2\pi i t} \) which has winding number 1, because \( 2\pi i t \) is a continuous choice of \( \arg(e^{2\pi i t}) \), hence \( \frac{1}{2\pi i}(\arg(e^{2\pi i 1}) - \arg(e^{2\pi i 0})) = 1 \). But this is the winding number of \( g(e^{2\pi i t}) \), contradicting the proposition.

**Proofs of the Fundamental Theorem of Algebra**

Every non-constant polynomial over \( \mathbb{C} \) has a root.

Let \( p \) be a polynomial \( p(z) = a_n z^n + \ldots + a_1 z + a_0 \), \( a_n \neq 0 \), \( n \geq 1 \). Let \( R \) be very big (how big will come later).

We look at the following closed paths from \( [0, 1] \to \mathbb{C} \).

\[
U(t) = p(R e^{2\pi i t}) \quad V(t) = a_n (R e^{2\pi i t})^n \quad W(t) = U(t) - V(t)
\]

Suppose that \( p \) has no zero, and suppose that \(|W(t)| < |V(t)|\)
Then the Dog Walking Lemma implies that the winding number of $u$
$w(u, 0) = w(v, 0)$.

However, the winding number since $2\pi$tn is a continuous choice of
$\arg(v(t))$. But then, by the earlier proposition about functions on $D$,
(generalised to functions on the disc of radius $R$), $p$ does have a root
$z$ with $|z| < R$, $\times$.

It remains to be shown that if $R$ is sufficiently large, then $|w(t)| < |v(t)|$. 
But $|w(t)| \leq (|a_0| + \ldots + |a_n|) R^{n-1}$ if $R > 1$, by the Triangle Inequality.

Hence everything works if $R > \frac{|a_0| + \ldots + |a_{n-1}|}{|a_n|}$, since
$|v(t)| = |a_n| R^n$.
Chapter 2: Approximation of Continuous Functions by Polynomials

Let $f: [a, b] \to \mathbb{R}$ be continuous. How closely can $f$ be approximated by polynomials? There is no loss of generality in considering $[0, 1].$

What do we mean by approximate? There are many ways to measure the "distance" between functions and hence to say how good our approximations are. E.g., $\|f - g\|_1 = \int_a^b |f(x) - g(x)| \, dx$ or $\|f - g\|_2 = \left( \int_a^b |f(x) - g(x)|^2 \, dx \right)^{1/2}.$ We will be concerned exclusively with the $\sup$-norm of $L^\infty,$ defined by $\|f\|_\infty = \sup_{x \in [0, 1]} |f(x)|.$

If $f$ is continuous, then this sup is also a maximum (recalling that a continuous function on a closed, bounded interval such as $[0, 1]$ is bounded and attains its bounds. Indeed, $f([0, 1])$ is sequentially compact, so is closed and bounded). For brevity, write $\|f\| = \|f\|_\infty.$

Recall that this gives a distance $d(f, g) = \|f - g\|.$ This gives the space $C[0, 1]$ of continuous real-valued functions the structure of a metric space, which turns out to be complete.

We will prove the following facts:

1. Every $f \in C[0, 1]$ can be uniformly approximated by polynomials. That is, for every $\varepsilon > 0,$ there is a polynomial such that $\|f - p\| < \varepsilon.$

Put another way, the polynomials are dense in $C[0, 1].$ This is a special case of the Stone-Weierstrass Approximation Theorem.

2. Among polynomials of degree at most $n,$ there is a unique best
approximation to \( f \), i.e. a polynomial \( P_n \) such that \( \| f - P_n \| < \| f - P \| \)
for all \( P \in P_n \) \( \leq \{ \text{polynomials of degree } \leq n \}, \ P \not\equiv P_n \).
We will say a few things about \( P_n \) while doing this.

**Theorem (Weierstrass Approximation Theorem)**

Let \( f \in C[0,1] \) be a continuous function, and \( \varepsilon > 0 \). Then there is a polynomial \( P \) with \( \| f - P \| \leq \varepsilon \).

**Proof (Bernstein)**

Let \( \mathbb{N} \) be a large integer (we will choose \( n \geq N_0(ef) \) depending on \( \varepsilon, f \)).
Let \( t \in [0,1] \). Let \( X_1, \ldots, X_n \) be independent Bernoulli random variables with mean \( t \). Then \( \mathbb{P}(X_i = 1) = t \), \( \mathbb{P}(X_i = 0) = 1 - t \).

Let \( Y_t = \frac{1}{n} \sum_{i=1}^{n} X_i \). We claim that \( \mathbb{E}(f(Y_t)) \) is a polynomial in \( t \), and that it closely approximates \( f(t) \). Idea: \( Y_t \) is highly concentrated about \( t \), so with high probability, \( f(Y_t) \approx f(t) \).

By the Law of Large Numbers, \( \mathbb{P}(X_1 + \cdots + X_n = k) = \binom{n}{k} t^k (1-t)^{n-k} \)

Thus \( \mathbb{E}(f(Y_t)) = \sum_{k=0}^{n} \mathbb{P}(Y_t = \frac{k}{n}) f\left(\frac{k}{n}\right) \)

\[= \sum_{k=0}^{n} \binom{n}{k} f\left(\frac{k}{n}\right) t^k (1-t)^{n-k} \]

is a polynomial in \( t \) of degree \( n \).

Now we need to show that \( P(t) = \mathbb{E}(f(Y_t)) \) approximates \( f(t) \)
densely for large \( n \). Let \( \varepsilon > 0 \). Choose \( N > 0 \) such that

\[|x - y| \leq \frac{\varepsilon}{2} \Rightarrow |f(x) - f(y)| \leq \frac{\varepsilon}{2} \] (possible since \( f \) is uniformly continuous).

Now \( |P(t) - f(t)| = |\mathbb{E}(f(Y_t)) - f(t)| \leq \frac{\varepsilon}{2} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \).
$E \left( |f(Y_e) - f(t)| \right) \leq \frac{\varepsilon}{2} + 2\|f\| \|P(1Y_e - t) \geq \delta\|

Explanation: If $|Y_e - t| \leq \delta$ then $|f(Y_e) - f(t)| \leq \frac{\varepsilon}{2}$, otherwise, the trivial bound $|f(Y_e) - f(t)| \leq 2\|f\| \|P\|$. To complete the proof, we need only show that if $n$ (dependent on $\varepsilon, f, \delta$) in large enough, then $P(1Y_e - t \geq \delta) \leq \frac{\varepsilon}{4} \|f\| \|P\|

We will actually show that $P(1Y_e - t \geq \delta) \leq \frac{1}{\delta^2 n}$ which is definitely sufficient.

This is proved using Chebyshev's Inequality (1A Probability).

Recall that we define, for a random variable $X$ with $E(X) = \mu, Var(X) = E((X-\mu)^2)$. Also, if $X_1, \ldots, X_n$ are independent, then $Var(X_1 + \ldots + X_n) = Var(X_1) + \ldots + Var(X_n)$ (by expanding $(*)$).

If $X_i$ is Bernoulli, meant, then $Var(X_i) = t(1-t)$.

Then $Var(Y_e) = Var(\frac{1}{n}(X_1 + \ldots + X_n)) = \frac{1}{n^2} Var(X_1 + \ldots + X_n) = \frac{1}{n} t(1-t) \leq \frac{1}{n} \quad \text{(since } t(1-t) \leq \frac{1}{4}, \quad t \in [0,1])$

Chebyshev's Inequality: $P(1X-\mu \geq \delta) \leq \frac{Var(X)}{\delta^2}$

(Proof $Var(X) = E((X-\mu)^2) \geq \delta^2 P(1X-\mu \geq \delta)$)

Putting this together gives the claimed bound. □

Task 2: Best Approximation

Writing $P_n = \{\text{polynomials of degree } \leq n\}$

Theorem

Let $f: [0,1] \to \mathbb{R}$ be continuous. Then for each $n$, the minimum
\[ \inf_{p \in \mathbb{R}^n} \| f - p \| \text{ is attained. (N.B. not necessarily uniquely attained)} \]

**Proof**

Consider \( \psi : \mathbb{R}^n \rightarrow \mathbb{R} \) defined by

\[ \psi(a_0, a_1, \ldots, a_n) = \max_{t \in [0, 1]} | f(t) - (a_0 + a_1 t + \ldots + a_n t^n) | = \| f - P_{a_0, a_1, \ldots, a_n} \| \]

where \( P_{a_0, a_1, \ldots, a_n} = a_0 + a_1 t + \ldots + a_n t^n \). We claim that \( \psi \) is continuous. Indeed,

\[ | P_{a_0, a_1, \ldots, a_n}(t) - P_{a_0', a_1', \ldots, a_n'}(t) | \leq | a_0 - a_0' | + \ldots + | a_n - a_n' | \]

by the triangle inequality.

Hence,

\[ | \psi(a_0, a_1, \ldots, a_n) - \psi(a_0', a_1', \ldots, a_n') | \leq | a_0 - a_0' | + \ldots + | a_n - a_n' | \]

\[ \leq (n+1) \| (a_0, a_1, \ldots, a_n) - (a_0', a_1', \ldots, a_n') \|_2 \]
Weierstrass

If \( f \in C([0,1]) \), \( \varepsilon > 0 \), then there is some polynomial \( p \) with \( \|f - p\| < \varepsilon \).

Recall

\( \mathbb{P}_n \) is the set of polynomials with degree \( \leq n \) and real coefficients

**Theorem**

\[ \inf_{p \in \mathbb{P}_n} \|f - p\| \text{ is attained, for every } f \in C([0,1]). \]

**Proof**

Consider \( \Psi : \mathbb{R}^n \to \mathbb{R} \) defined by \( \Psi(a_0, \ldots, a_n) = \|f - P_{a_0, \ldots, a_n}\| \)
where \( P_{a_0, \ldots, a_n}(x) = a_0 + a_1 x + \ldots + a_n x^n \). We showed that \( \Psi \) is continuous. Now observe that there is some \( c > 0 \) such that \( \|P_{a_0, \ldots, a_n}\| \geq c \) uniformly for all \( (a_0, \ldots, a_n) \) on the unit sphere in \( \mathbb{R}^n \). Indeed, \( (a_0, \ldots, a_n) \to \|P_{a_0, \ldots, a_n}\| \) is continuous and the unit sphere is compact. Furthermore, \( 0 \) does not lie in the image of \( \Psi \), since only the zero polynomial \( P_{0, \ldots, 0} \) vanishes on \([0,1]\).

Note, however, that \( \|P_{a_0, \ldots, a_n}\| = \sum \|P_{a_0, \ldots, a_n}\| \) and hence \( \|P_{a_0, \ldots, a_n}\| \to \infty \) as \( (a_0, \ldots, a_n) \to \infty \). It follows that there is some \( R \) such that \( \|f - P_{a_0, \ldots, a_n}\| \geq \) \( \varepsilon \) \( \inf_{p \in \mathbb{P}_n} \|f - p\| \)
whenever \( \| (a_0, \ldots, a_n) \| \geq R \). Indeed \( \frac{1}{c} \to \infty \) as \( R \to \infty \), and \( \frac{1}{c} \) is finite. Hence, \( \inf_{p \in \mathbb{P}_n} \|f - p\| = \inf_{\| (a_0, \ldots, a_n) \| \leq R} \|f - P_{a_0, \ldots, a_n}\| \)
which is the infimum of a continuous function on a compact set and is thus attained.
Finding the Closest Polynomial to \( f \) (best approximation)

How do we recognize that \( p \) is the best approximation to \( f \), with \( p \in P_n \)?

Certainly, if \( F := f - p \) has the property that \( \|F + q\| \geq \|F\| \) whenever \( q \in P_n \). Such a function \( F \) shall be referred to as \( n \)-unimprovable.

**Theorem (Chebyshev's Equal-Ripple Theorem)**

A continuous function \( F : [0, 1] \to \mathbb{R} \) is \( n \)-unimprovable \( \iff \) there exist \( n + 2 \) nodes \( 0 \leq a_0 < a_1 < \ldots < a_{n+1} \leq 1 \) such that

\[
F(a_i) = \pm \|F\| \text{ with alternating signs.}
\]

**Proof**

The easier direction is 'if', \( \Rightarrow \). Suppose that there are nodes \( a_0, \ldots, a_{n+1} \). Suppose that \( q \in P_n \) is such that \( \|F + q\| < \|F\| \).

Suppose that \( F(a_i) = (-1)^i \|F\| \) (the other case is identical).

Then \( q(a_0) < 0 \) (otherwise \( (F+q)(a_0) > \|F\| \)), and \( q(a_1) > 0 \), \( q(a_2) < 0 \), and so on. So by the Intermediate Value Theorem, \( q \) has at least \( n+1 \) roots, one in each interval \( (a_i, a_{i+1}) \). This is a contradiction as \( q \in P_n \).

The 'only if', \( \Leftarrow \) direction is more tricky.

Let \( F \) be an \( n \)-unimprovable function. Define \( a_0 \) to be the first point where \( |F| \) first attains its maximum. WLOG, \( F(a_0) = \|F\| \). Let \( a_1 \) be the first point after \( a_0 \) with \( F(a_1) = -\|F\| \), then define \( a_2, a_3, \ldots \) and so on. We get points \( a_0, \ldots, a_m \). We must prove that \( m \geq n+1 \).
Suppose instead that \( m \leq n \). We will show that \( \exists q \in \mathbb{P}_n \) such that \( \|F + q\| < \|F\| \), so \( F \) wasn't \( n \)-unimprovable after all.

Let \( z_i \) be the last point in \( (a_{i-1}, q_i) \) where \( F(z_i) = 0 \).

Take \( q(x) = c (x - z_1) \cdots (x - z_m) \) for some very small \( c \).

Note that \( q \in \mathbb{P}_n \). Choose the sign \( \pm \) to match the diagram.

We claim that for sufficiently small \( c \), \( \|F + q\| < \|F\| \).

Certainly everything is fine at the extrema \( a_i \), since the signs were chosen to make this so. Things could go wrong at points such as \( b \), but we must have \( F(b) > F(a_i) + \Delta \) (1), \( \Delta > 0 \), otherwise we would have chosen \( b \) instead of \( a_i \). Choosing \( c \) small enough, we have \( (F+q)(b) > F(a_i) \).

(1) N.B. On \([a_0, z_1]\), \( F \) attains its minimum, which is not \( F(a_i) \).

**Theorem**

Suppose that \( f \in C[0,1] \). Then there is a unique best approximation to \( f \) in \( \mathbb{P}_n \).

**Proof**

Suppose that \( f - p \) is a best approximation to \( f \). Then \( F \) is \( n \)-unimprovable. Thus \( \|F + q\| > \|F\| \). We claim that this is a strict inequality if \( q \neq 0 \). Suppose that \( \|F + q\| \leq \|F\| \).

By the triangle inequality, \( \|F + \frac{q}{2}\| \leq \frac{1}{2} \|F\| + \frac{1}{2} \|F + q\| = \|F\| \).

So \( F + \frac{q}{2} \) is also \( n \)-unimprovable.
By the equal ripple criterion (difficult direction) there are nodes 

$$a, \ldots, a_n$$ with the claimed properties.

Then $$|F + \frac{q}{2_i}(a_i)| = ||F||$$ for each $$i$$.

But $$|F + \frac{q}{2_i}(a_i)| \leq \frac{1}{2} |F(a_i)| + \frac{1}{2} |F + q_i(a_i)|$$ (triangle inequality)

Equality must occur, so $$F + \frac{q}{2_i}(a_i) = F(a_i) = (F + q_i)(a_i)$$.

Hence $$q_1(a_0) = \ldots = q_{i+1}(a_{n+1}) = 0$$. But $$q \in P_n$$, so $$q \equiv 0$$.
Polynomial Interpolation

Lagrange Interpolation

Given \( n+1 \) points \( a_0, \ldots, a_n \in \mathbb{R} \), and some function \( f: \mathbb{R} \to \mathbb{R} \). Then we may construct explicitly a polynomial \( p \) of degree \( \leq n \) with \( f(a_i) = p(a_i), \ 0 \leq i \leq n \). Indeed, consider \( L_i(x) = \prod_{j \neq i} \frac{x - a_j}{a_i - a_j} \)

Note that \( L_i(a_j) = \begin{cases} 1 & i = j \\ 0 & \text{otherwise} \end{cases} \). Take \( p(x) = \sum_{i=0}^{n} f(a_i) L_i(x) \)

It is pretty obvious that \( p \) has the stated properties.

Basic Question

Is the polynomial \( p \) constructed in this way a good approximation to \( f \) in the uniform norm?

Answer 1

No, not for any choice of the \( a_i \), for particularly badly chosen \( f \)

Answer 2

If \( f \) is "nice" then this choice \( p \) may be a good approximation to \( f \), but some choices of the nodes \( a_i \) are better than others. In particular, equally spaced nodes are not great.

Theorem

Suppose \( a_0, \ldots, a_n \) are distinct points in \([-1, 1]\). Suppose that \( f: \mathbb{R} \to \mathbb{R} \) is infinitely differentiable on some interval containing \([-1, 1]\).

Let \( p \) be the Lagrange Interpolant to \( f \) of degree \( \leq n \); that is to say the unique \( p \) with degree \( \leq n \) and \( f(a_i) = p(a_i), 0 \leq i \leq n \).
Then for each \( x \in (-1, 1) \), there is a \( \xi \in (-1, 1) \) such that
\[
\frac{f(x) - p(x)}{w(x)} = \frac{1}{(n+1)!} \int_{a_n}^{a_0} f^{(n+1)}(\xi)(x-a_n) \ldots (x-a_0) d\xi
\]
In particular \( \| f - p \| \leq \frac{1}{(n+1)!} \| f^{(n+1)} \| \| (x-a_n) \ldots (x-a_0) \|_{\infty} \).

**Proof**

Define \( r(t) = f(t) - p(t) \), and \( w(t) = (t-a_n) \ldots (t-a_0) \),
\( y(t) = r(t) - \frac{w(t)}{w(x)} r(x) \). Note that \( y(x) = 0 \) (indeed, \( w(a_i) \neq 0 \) and \( r(a_i) = f(a_i) - p(a_i) = 0 \)). Suppose that \( x \notin \{a_0, ..., a_n \} \) (otherwise the result is trivial). Recall Rolle's Theorem:

If \( f \) is differentiable and \( f(a) = f(b) = 0 \), then there is \( c \in (a, b) \) such that \( f'(c) = 0 \). We have a function \( y \), with \( n+2 \) distinct zeroes.

Applying Rolle \( n+1 \) times, we get a \( \xi \) where \( y^{(n+1)}(\xi) = 0 \).

Now \( w^{(n+1)}(\xi) = (n+1)! \), since \( w \) is a monic polynomial of degree \( n+1 \).

Also, \( f^{(n+1)}(\xi) = f^{(n+1)}(\xi) \), since \( r = f - p \), and \( p \) is a polynomial of degree \( \leq n \). Hence \( 0 = y^{(n+1)}(\xi) = f^{(n+1)}(\xi) - \frac{(n+1)! f(x)}{w(x)} \).

This is what was claimed.

**Remark**

When \( n = 0 \), this is the Mean Value Theorem.

**Problem**

How should we choose the nodes \( a_0, ..., a_n \) to minimize \( \max_{x \in (-1, 1)} \| (x-a_n) \ldots (x-a_0) \|_{\infty} \), and hence, in view of the Theorem, give a good approximation to \( f \).
Chebyshev Polynomials

Definition

$T_n$ is the unique polynomial (of degree $n$) such that $T_n(\cos \theta) = \cos n\theta$ for all $\theta$. We can calculate the first few by hand:

$\cos 2\theta = 2\cos^2 \theta - 1$, $T_2(x) = 2x^2 - 1$

$\cos 3\theta = 4\cos^3 \theta - 3\cos \theta$, $T_3(x) = 4x^3 - 3x$

1. $\cos((n+1)\theta) = \cos n\theta \cos \theta - \sin n\theta \sin \theta$

2. $\cos((n+2)\theta) = \cos n\theta \cos 2\theta - \sin n\theta \sin 2\theta = \cos n\theta \cos 2\theta - 2\sin n\theta \sin \theta \cos \theta$

$-2\sin 2\cos \theta \Rightarrow \cos((n+2)\theta) - 2\cos((n+1)\theta) \cos \theta = -\cos n\theta$

Hence, if we know that $T_n$, $T_{n+1}$ exist then so does $T_{n+2}$, and

$T_{n+2}(x) = 2xT_{n+1}(x) - T_n(x)$. So the Chebyshev polynomials are easy to compute and $T_n(x) = 2^{n-1}x^n + \cdots$

Note that for $x \in [-1,1]$, $|T_n(x)| \leq 1$, and the maximum is attained when $x = \cos \theta$, $\theta = \frac{\kappa \pi}{n}$, $\kappa = 0, 1, \ldots, n$. Write

$q_k = \cos \frac{\kappa \pi}{n}$ (these are called the Chebyshev nodes). Then $T_n(q_k) = (-1)^k$

This means that $T_n$ is $(n-1)$-unimodal, which means that

$\|T_n + p\| \geq \|T_n\|_{L^\infty([-1,1])}$ whenever $p$ is a polynomial of degree $\leq n-1$.

Equivalently, $\frac{1}{2^n-1} T_n(x)$ has the smallest supremum norm on $[-1,1]$, amongst all polynomials of degree $\leq n$. Changing $n$ to $n+1$, we see that

$\inf_{x \in [-1,1]} |(x-a_0)(x-a_n)| \leq \frac{1}{2^n}$ with equality
when $b_0, \ldots, b_n$ are the roots of the Chebyshev polynomial $T_{n+1}(x)$. These are the Chebyshev nodes. They are given by $b_k = \cos\left(\frac{(k+1)\pi}{n+1}\right)$.

These give a good set of points for polynomial interpolation.

**Runge's Phenomenon**

With equally spaced interpolation points $a_0, \ldots, a_n$, it is possible to have $\|f - p_n\| \to \infty$, where $p_n$ is the degree $\leq n$ interpolant to $f$ at the points $a_0, \ldots, a_n$, even for "nice" $f$. Runge's example is $f(x) = \frac{1}{1 + 25x^2}$ on $[-1, 1]$. For $x = 0.9$ (say), $|f(x) - p_n(x)| \to \infty$.

Interpolating at the Chebyshev nodes is generally, but not always, more successful.
Quadature and Legendre Polynomials

Problem

Given a continuous function $f : [-1, 1] \to \mathbb{R}$, approximate the integral $\int_{-1}^{1} f(t) \, dt$ by a discrete sum $\sum_{i=1}^{n} \lambda_i \cdot f(a_i)$ where the $\lambda_i \in \mathbb{R}$ are some weights and $-1 < a_1 < a_2 < \ldots < a_n < 1$, are nodes.

We will show that by choosing $\lambda_i$, judiciously, this can be done.

It turns out that choosing $\lambda_i = \frac{1}{n}$, $a_i$ equally spaced in not a good choice.

Lemma

Given nodes $a_1, \ldots, a_n$, there is a (unique) choice of weights $\lambda_i$ such that the "quadrature formula" $\int_{-1}^{1} f(t) \, dt = \sum_{i=1}^{n} \lambda_i f(a_i)$ whenever $f \in \mathbb{P}_n$, the space of polynomials of degree $\leq n$.

Proof (Linear Algebra)

The map which sends $f$ to $\int_{-1}^{1} f(t) \, dt$ is a linear functional on the space $V = \mathbb{P}_n$, that is to say it is in $V^*$. It suffices to show that the evaluation maps $f \mapsto f(a_i)$ give a basis for $V^*$.

Since $\dim V^* = \dim V = n$, it suffices to show that these are linearly independent i.e. $\sum_{i=1}^{n} \lambda_i f(a_i)$ for all $f \in V = \mathbb{P}_n$.

$\Rightarrow \lambda_1 = \ldots = \lambda_n = 0$

Proof

Substitute $f(x) = x^i$, $i = 0, \ldots, n-1$. We get
\[
\begin{pmatrix}
\alpha_1 & \cdots & \alpha_n \\
\alpha_1^2 & \cdots & \alpha_n^2 \\
\vdots & \ddots & \vdots \\
\alpha_1^n & \cdots & \alpha_n^n
\end{pmatrix}
\begin{pmatrix}
\beta_1 \\
\beta_2 \\
\vdots \\
\beta_n
\end{pmatrix} = 0.
\]

But the determinant of the matrix is \( \prod_{i \neq j} (\alpha_i - \alpha_j) \neq 0 \) (Vandermonde determinant).

Hence, \( \lambda_1 = \lambda_2 = \ldots = \lambda_n = 0 \).

**Proof 2**

Construct \( f \) such that \( f(\alpha_i) = 1 \) and \( f(\alpha_j) = 0 \) for \( j \neq i \), by Lagrange interpolation i.e.

\[
f(x) = \prod_{i \neq j} \frac{x - \alpha_i}{\alpha_i - \alpha_j} = L_i(x)
\]

The second proof gives a formula for the weights \( \lambda_i \). Indeed, substituting \( f(t) = L_i(t) \) into the quadrature formula gives

\[
\lambda_i = \int_{-1}^{1} L_i(t) \, dt.
\]

Thus, for any choice of the nodes \( \alpha_i \), we have a "reasonably sensible" choice of the weights \( \alpha \). It turns out that there is a particularly good choice of the \( \alpha_i \) coming from the Legendre polynomials.

**Legendre Polynomials**

**Theorem**

There are polynomials \( p_n \), unique up to scalar multiple, such that \( \deg p_n = n \) and \( \int_{-1}^{1} p_m(t) p_n(t) \, dt = 0 \) if \( m \neq n \).

If the scalar multiple is chosen so that \( p_n(1) = 1 \), these are called the Legendre Polynomials.

**Proof**

Apply the Gram-Schmidt process. Suppose that \( p_0, p_1, \ldots, p_{n-1} \) have already been constructed. We can take \( p_0(x) = 1, p_1(x) = x \).
To make $p_n$, take a polynomial $F$ of degree $n$ (say $F(x) = x^n$) and define $p_n(x) = \sum_{i=0}^{n-1} \langle F, p_i \rangle p_i$, where $\langle f, g \rangle$ denotes the inner product $\int_{-1}^{1} f(t)g(t) \, dt$.

Then $\langle p_n, p_m \rangle = \langle F, p_m \rangle - \sum_{i=0}^{n-1} \langle F, p_i \rangle \langle p_i, p_m \rangle$

$= \langle F, p_m \rangle - \langle F, p_m \rangle = 0.$

Note that $p_n$ has degree $n$. Why are the $p_i$ unique?

The polynomials $p_0, \ldots, p_{n-1}$ are linearly independent (either using the fact that orthogonal vectors are always linearly independent, or alternating supposing that $\lambda_0p_0 + \cdots + \lambda_{n-1}p_{n-1} = 0$, and considering the coefficients of $x^{n-1}$ to get $\lambda_{n-1} = 0$, inducting downwards). Thus we must have $p_n = F - \sum_{i=0}^{n-1} \lambda_i p_i$. Now, taking inner products with $p_m$ gives $0 = \langle F, p_m \rangle - \lambda_m \langle p_m, p_m \rangle$.

So indeed $\lambda_i = \frac{\langle F, p_i \rangle}{\langle p_i, p_i \rangle}$.

**Lemma**

$p_n(x)$ has distinct roots $a_1, \ldots, a_n$ in $[-1, 1]$.

Suppose not. Let $a_1, \ldots, a_k$ be all of the crossing roots of $p_n$. Suppose $k < n$, and consider $q_n(x) = (x - a_1) \ldots (x - a_k)$. Then $q_n p_n$ is always positive or always negative on $[-1, 1]$, and in particular $\int_{-1}^{1} q_n p_n(t) \, dt \neq 0$. But $p_n$ is orthogonal to all polynomials (such as $q_n$) of degree $< n$.

Thus $p_n$ has $n$ distinct roots $a_1, \ldots, a_n$; the Legendre nodes.
Consider the quadrature formula attempt \( \int_{-1}^{1} f(t) \, dt \approx \sum_{i=1}^{n} \lambda_i f(a_i) \)
with these nodes, and the \( \lambda_i \) chosen so that it is an exact formula for \( f \in P_{2n} \).

**Claim**

The quadrature formula is in fact exact for all \( f \in P_{2n} \), i.e., for all polynomials of degree \( 2n-1 \).

**Proof**

Given a polynomial \( f \in P_{2n} \), we can write \( f(x) = p_n(x)Q(x) + r(x) \) with \( \deg Q \leq n-1 \), \( \deg r \leq n-1 \). Then we have
\[
\int_{-1}^{1} f(t) \, dt = \int_{-1}^{1} r(t) \, dt \quad \text{(since } \langle p_n, Q \rangle = 0 \text{, orthogonality)}
\]
Also \( \sum_{i=1}^{n} \lambda_i f(a_i) = \sum_{i=1}^{n} \lambda_i (p_n(a_i)Q(a_i) + r(a_i)) = \sum_{i=1}^{n} \lambda_i r(a_i) \)
(since \( p_n(a_i) = 0 \)). But \( \int_{-1}^{1} r(t) \, dt = \sum_{i=1}^{n} \lambda_i r(a_i) \)
nice \( \deg r \leq n-1 \).

**Lemma**

If the \( \lambda_i \) are the weights attached to the Legendre nodes \( a_1, \ldots, a_n \), then \( \sum_{i=1}^{n} \lambda_i = 2 \) and \( \lambda_i > 0 \) for all \( i \) (a crucial point).

**Proof**

For the first part, substitute \( f = 1 \) into the quadrature formula (works for arbitrary nodes). For the second statement, \( f(x) = \prod_{j \neq i} (x - a_j)^2 \) has degree \( 2n-2 \). Thus \( 0 < \int_{-1}^{1} f(t) \, dt = \sum_{i=1}^{n} \lambda_i f(a_i) = \prod_{j \neq i}(a_i - a_j)^2 \lambda_i \).
Hence \( \lambda_i > 0 \).
Legendre nodes $a_i$, $-1 \leq a_1 < a_2 < \ldots < a_n \leq 1$

$$\int_{-1}^{1} f(t) \, dt \approx \sum_{i=1}^{n} \lambda_i f(a_i)$$

exact for deg $f < n$

When the $a_i$ are the Legendre nodes, exact for deg $f < 2n$

In this case, $\sum_{i=1}^{n} \lambda_i = 2$, $\lambda_i > 0$.

**Proposition**

Let $f: [-1, 1] \rightarrow \mathbb{R}$ be a continuous function. Then, as $n \rightarrow \infty$,

$$\sum_{i=1}^{n} \lambda_i f(a_i) \rightarrow \int_{-1}^{1} f(t) \, dt$$

**Proof**

This is obvious when $f$ is a polynomial, since the formula is exact when $n > \text{deg } f$. In the general case, let $\varepsilon > 0$.

By the Weierstrass Approximation Theorem, there is a polynomial $\hat{f}$ such that $\|f - \hat{f}\| < \varepsilon/2$. We have

$$\left| \int_{-1}^{1} f(t) \, dt - \int_{-1}^{1} \hat{f}(t) \, dt \right| < \frac{\varepsilon}{2} \quad (i)$$

$$\sum_{i=1}^{n} \lambda_i f(a_i) = \sum_{i=1}^{n} \hat{f}(a_i) \quad \text{for sufficiently large } n \quad (ii)$$

$$\left| \sum_{i=1}^{n} \lambda_i f(a_i) - \sum_{i=1}^{n} \lambda_i \hat{f}(a_i) \right| \leq \sum_{i=1}^{n} |\lambda_i| |f(a_i) - \hat{f}(a_i)| \quad (iii)$$

$$< \frac{\varepsilon}{2} \sum_{i=1}^{n} |\lambda_i| = \frac{\varepsilon}{2} \quad \text{since } \lambda_i > 0, \sum_{i=1}^{n} \lambda_i = 2$$

Combining $i)$, $ii)$, $iii)$, and using the Triangle Inequality gives

$$\left| \sum_{i=1}^{n} \lambda_i f(a_i) - \int_{-1}^{1} f(t) \, dt \right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad \text{for sufficiently large } n$$

**Remark**

The positivity of the $\lambda_i$ was crucial. When the $a_i$ are equally spaced, some of the $\lambda_i$ are negative, at least for $n \geq 8$.

Indeed, it turns out that there are continuous $f$ for which the equally spaced quadrature formula diverges.
Remark
On Example Sheet 3:

i) Compute a few Legendre Polynomials

ii) Recurrence relation linking $P_n$, $P_{n+1}$, $P_n$

iii) Rodrigue's Formula: $P_n(x) = \frac{1}{2^n n!} \left( \frac{d^n}{dx^n} \right)^n (1-x^2)^n$

(Hint: prove $\int x^m P_n(x) P_m(x) \, dx = 0$ by repeated integration by parts, then check $P_n(1) = 1$)

Approximation by complex polynomials

Basic Question: Suppose that $K \subseteq \mathbb{C}$ is compact, and $f: K \to \mathbb{C}$ continuous. Can $f$ be uniformly approximated by polynomials on $K$? That is, is it possible true that for every $\varepsilon > 0$, there is a polynomial $p: \mathbb{C} \to \mathbb{C}$ such that $\sup_{z \in K} |p(z) - f(z)| \leq \varepsilon$?

(Complex version of Weierstrass approximation)

There are two basic obstructions to such a result being true.

1. The uniform limit of holomorphic (complex differentiable functions) is holomorphic. More precisely, if $D \subseteq \mathbb{C}$ is an open ball, and if $f_n \to f$ uniformly on $D$, $f_n$ holomorphic, then $f$ is holomorphic.

Recall the proof:

Morera's Theorem says that a continuous function $g: D \to \mathbb{C}$ is holomorphic $\iff$ $\int_T g(z) \, dz = 0$ for every triangle $T \subseteq D$.

One direction is via Cauchy's Theorem.

Conversely, define $G(z) = \int_{[z_0 \to z]} g(w) \, dw$. Then $G(z)$ is differentiable with derivative $g(z)$. Indeed,

$$
\frac{G(z+h) - G(z)}{h} = \frac{1}{h} \int_{[z \to z+h]} g(w) \, dw \xrightarrow{h \to 0} \frac{1}{k} \int_{[z \to z+h]} g(z) \, dw = g(z)
$$

But holomorphic functions are infinitely differentiable, and so $g(z)$ is holomorphic. (Prove that something is differentiable by integrating it!!)
Returning to our discussion of obstruction 1, suppose that $\partial T$ is the boundary of a triangle $T$. Then

$$| \int_{\partial T} f_n(z) \, dz - \int_{\partial T} f(z) \, dz | \leq \text{length}(\partial T) \| f_n - f \|_{L^\infty} \to 0$$

since $f_n \to f$ uniformly. But $\int_{\partial T} f_n(z) \, dz = 0 \forall n$, so $\int_{\partial T} f(z) \, dz = 0$. Hence $f$ is holomorphic. Note that $f$ was certainly continuous as the uniform limit of continuous functions.

2 "Topological Obstructions". Let $k = \{ z \in \mathbb{C} | |z| \leq 2 \frac{3}{4} \}$.

$f(z) = \frac{z}{3}$. $f(z)$ is holomorphic on $k$. However, $f$ is not the uniform limit of polynomials.

Indeed, let $r$ be the contour $r(t) = e^{2\pi it}$, $t \in [0, 1]$.

If $p$ is a polynomial, $\int_{r} p(z) \, dz = 0$ (Cauchy's Theorem).

However, $\int_{r} \frac{1}{z} \, dz = \int_{1}^{e^{2\pi i}} \frac{r'(t)}{r(t)} \, dt = 2\pi i$.

But $\int_{r} (p(z) - \frac{1}{z}) \, dz \leq \text{length}(r) \| p - \frac{1}{z} \|_{L^\infty}$. and so $\| p - \frac{1}{z} \|_{L^\infty} \geq 1$.

More generally, there can be no "holes" in $f$.

In a sense, these are the only ways in which a function $f : k \to \mathbb{C}$ can fail to be uniformly approximated by polynomials. This is called Runge's Theorem.

**Theorem**

Suppose that $k$ is compact and that $S$ is a domain, open in $\mathbb{C}$, such that $\mathbb{C} \setminus S$ is path connected. Then, let $f : S \to \mathbb{C}$ be holomorphic. Then $f$ can be uniformly approximated...
Path connected means that any two points in $\mathbb{C} \setminus \partial$ are joined by a path entirely in $\mathbb{C} \setminus \partial$.

**Broad Strategy**

1. $f$ can be uniformly approximated by rational functions, all of whose poles lie outside of $\partial$.

Then we will reduce to the case $f(z) = \frac{1}{a - z}$, $a \notin \partial$.

2. Handle $f(z) = \frac{1}{a - z}$, $a \notin \partial$. 


Riesz's Theorem

Theorem

Suppose that $K \subseteq \mathbb{C}$ is compact, $\Omega \subseteq \mathbb{C}$ open, and $K \subseteq \mathbb{R}$.
Suppose that $f : \mathbb{R} \to \mathbb{C}$ is analytic, and $\Omega \setminus K$ path connected.
Then $f$ may be uniformly approximated by polynomials.

"Can be uniformly approximated by polynomials on $K" \iff "Is UAP"

We observe that if $f, g : K \to \mathbb{C}$ are UAP then so are $f + g$, $fg$, and $\lambda f$, for any scalar $\lambda$.
(Proof on the Example Sheet. The key point is that $f$, $g$ are bounded on $K$).

Proof of Step 1:
Recall Cauchy's Integral Formula. If $f$ is holomorphic on a square domain $D$, and if $\gamma$ is a small square contour then
$$f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{z - w} \, dw$$
wherever $z \in \gamma$.

(Brief reminder of proof: Shrink the contour to a very small circular contour about $z$, $C_{\epsilon}(z)$. If $\epsilon$ is very small then $f(w) \approx f(z)$. The integral does not change.)

Since $K$ is compact, and $\Omega \setminus K$ is closed, there is some $\delta > 0$ such that $|x - y| > \delta$ whenever $x \in K$, $y \in \Omega$ (Sheet 1, Question 2).

Place on the complex plane a square grid, sidelength $\frac{\delta}{100}$, centred on 0. Apply Cauchy's Integral Formula around every subsquare in the grid which touches $K$.
Write $c_1, c_2, \ldots, c_m$ for all of the square contours. Provided that $z$ does not lie on
any $c_j$, we have \( f(z) = \frac{1}{2\pi i} \sum_{j=1}^{m} \int_{c_j} \frac{f(w)}{w - z} \, dw \).

Why? If $z$ lies inside $c_j$, apply Cauchy's Integral Formula. If $z$ does not lie in $c_j$, then \( \int_{c_j} f(w) \, dw = 0 \), since \( \frac{f(w)}{w - z} \) is holomorphic on an open domain containing $c_j$.

But every $z \in \mathbb{K}$ does lie in or on the boundary of one of the $c_j$.

When the sum over $j$ is taken, a lot of cancellation occurs. Any edge completely in $\mathbb{K}$ will be cancelled out.

Thus we obtain \( f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{w - z} \, dw \), where $\gamma$ is some closed contour lying between $\mathbb{K}$ and $\mathbb{C} \setminus \mathbb{R}$, made up of horizontal or vertical segments.

Parametrize this path $\gamma$ as a continuous path $\Phi : [0, 1] \to \mathbb{C}$.

Then we have \( f(z) = \frac{1}{2\pi i} \int_{0}^{1} \frac{f(\Phi(t))}{\Phi(t) - z} \, d\Phi(t) \) (definition of a path integral). Approximate the integral by a sum at the points $t = \frac{i}{N}, s = 0, 1, \ldots, N-1$ for some large $N$.

Write this as $S_{N}(z) = \frac{1}{2\pi i} \sum_{s=1}^{N} \frac{f(\Phi(\frac{s}{N}))}{\Phi(\frac{s}{N}) - z} \Phi'(\frac{s}{N})$. We will show in just a moment that $S_{N}(z) \to f(z)$ uniformly for $z \in \mathbb{K}$. This completes the proof of Step 1, since the sum consists of rational functions with poles outside $\mathbb{K}$. (The poles are at $\Phi(\frac{s}{N})$, which lie on $\gamma$, and hence outside $\mathbb{K}$.)

Why does $S_{N}(z) \to f(z)$ uniformly? Suppose that $F : [0, 1] \to \mathbb{R}$ is continuous. Then \( \left| \int_{0}^{1} F(t) \, dt - \frac{1}{N} \sum_{s=1}^{N} F(\frac{s}{N}) \right| \leq \sup_{x, y \in \mathbb{K}} |F(x) - F(y)| \to 0 \), as $F$ is uniformly continuous.

In our case, $F(t) = \frac{f(\Phi(t))}{\Phi(t) - z} \Phi'(t)$.
However, \( \lim_{|y| \to 0} |F_2(x) - F_2(y)| = 0 \) uniformly in \( \xi \), since the function \((z,t) \mapsto F_2(t)\) is uniformly continuous on \( K \times [0,1] \), since \( K \times [0,1] \) are compact, and the map is continuous because \( \varphi(t) \in \mathcal{K}\).

When \( \xi \) lies on a square \( C_i \). We get the formula
\[
\varphi(z) = \frac{1}{2\pi i} \int_{C_i} \frac{\varphi(w)}{w-z} \, dw, \quad \text{when } \xi \in K \text{ does not lie on the boundary of } C_i.
\]
However, both sides are continuous functions of \( \xi \) and so the formula also holds when \( \xi \) lies on the boundary of one of the \( C_i \).

**Step 2**

We now want to show that if \( \xi \notin K \), then \( \frac{1}{a-\xi} \) is UAP on \( K \).

If we can do this, then the proof of Runge is complete.

The idea is to consider the set \( S \) of \( a \) for which this is true.

We will show i) that \( S \) is non-empty (easy). Then ii) if \( a \in S \), and \( b \) is "close" to \( a \), then \( b \in S \). We will conclude iii) that \( S = C \setminus K \).

**Proof of i)**

Take \( a \) with \( |a| > 2 \max_{z \in K} |z| \). Then
\[
\frac{1}{a-\xi} = \frac{1}{a} \left( \frac{1}{1-\frac{\xi}{a}} \right) = \frac{1}{a} \left( 1 + \frac{\xi}{a} + \frac{\xi^2}{a^2} + \ldots \right)
\]

Series converges uniformly on \( \xi \in K \), since
\[
\sum_{n=0}^{\infty} \left| \frac{\xi^n}{a^n} \right| \leq \sum_{n=0}^{\infty} \frac{1}{2^n} = \frac{1}{2^n}
\]
Runge's Theorem (Second part of the proof)

$k \subset \mathbb{C}$ compact, $\mathbb{C} \setminus k$ path connected. Show that $\frac{1}{z-\alpha}$ is UAP if $\alpha \notin k$.

**Strategy**

$S = \{ \alpha : \frac{1}{z-\alpha} \text{ is UAP} \}$

i) $S$ non-empty (done last time)

ii) If $\alpha \in S$, and $\beta$ is close to $\alpha$, then $\beta \in S$.

iii) $S$ is all of $\mathbb{C} \setminus k$.

ii) Note that $\frac{1}{\beta - z} = \frac{1}{\alpha - z} \left( 1 + \frac{\alpha - \beta}{\alpha - z} + \left( \frac{\alpha - \beta}{\alpha - z} \right)^2 + \ldots \right)$.

If $|\alpha - \beta| < |\alpha - z|$, certainly if $|\alpha - \beta| < \text{dist}(\alpha, k) = \inf_{z \in k} |\alpha - z|$, then the series converges uniformly for $z \in k$ (use the triangle inequality).

Write $S_n(z) = \frac{1}{\alpha - z} \left( 1 + \frac{\alpha - \beta}{\alpha - z} + \ldots + \left( \frac{\alpha - \beta}{\alpha - z} \right)^n \right)$. By earlier remarks, $S_n(z)$ is UAP, since, by assumption, $\frac{1}{\alpha - z}$ is UAP, and $S_n(z)$ can be built up using just sums, products, and scalar multiples.

Now $S_n(z) \to \frac{1}{\beta - z}$ uniformly for $z \in k$.

(Indeed $|S_n(z) - \frac{1}{\beta - z}| \leq \frac{|\alpha - \beta|}{|\alpha - z|^2} \sum \frac{1}{n^2} \frac{1}{|\alpha - z|^2} \leq \frac{1}{\text{dist}(\alpha, k)} \sum \frac{1}{n^2}$

(Where $r = \max(1, |\alpha - z|) < 1$ $\Rightarrow 0$ as $N \to \infty$)

Finally, note that a uniform limit of UAP functions is UAP.

(Proof: If $f_n \to f$ uniformly, choose $p_n$ such that
We have established the following precise version of ii):

If \( x \in S \), \( |β - α| < \text{dist}(α, k) \), then \( β \in S' \).

iii) Suppose that \( β \in C \backslash k \) is arbitrary.

Let \( x \in S' \) (such an \( x \) exists by i)). Consider a continuous path from \( x \) to \( β \), that is, to say, a continuous map \( \varphi : [0, 1] \rightarrow \mathbb{C} \) with \( \varphi(0) = x \), \( \varphi(1) = β \). The image \( \varphi([0, 1]) \) is compact, since \( [0, 1] \) is compact, and hence closed. Hence \( (\text{Sheet1}, \text{Q2}) \), \( \exists ϵ > 0 \) such that \( \text{dist}(\varphi(t), k) > ϵ \) uniformly for \( t \in [0, 1] \).

Taking \( ϵ \) from iii) one finds a sequence \( 0 = t₀ < t₁ < \ldots < tₘ = 1 \), with \( t_{i+1} - t_i < ϵ \) for each \( i \). Then, for each \( i \), \( \text{dist}(\varphi(tᵢ), \varphi(t_{i+1})) < ϵ \leq \text{dist}(\varphi(tᵢ), k) \). Hence by ii), \( \varphi(tᵢ) \in S' \Rightarrow \varphi(tᵢ) \in S' \). Hence, since \( \varphi(t₀) = x \in S' \), we know \( \varphi(tₘ) = β \in S' \).

**Theorem**

There exists a sequence of polynomials \( pₙ \) converging uniformly to a discontinuous function on \( \mathbb{C} \), \( |z| ≤ 1 \).

**Proof**

Consider the following compact sets \( kₙ \).
Topics in Analysis (12)

\[ \arg z = \frac{2\pi}{n} \]

\[ \mathbb{C} \quad \text{On } \mathbb{C}, \text{ we can define a continuous branch of } \sqrt{z} \]

\[ f_n(r e^{i\Theta}) = r^{\frac{1}{n}} e^{i\frac{\Theta}{n}} \text{ where } \frac{2\pi}{n} \leq \Theta \leq 2\pi + \frac{2\pi}{n} \]

This is holomorphic on an open set \( \Omega \), with \( \mathbb{C} \setminus \delta \), and \( \mathbb{C} \setminus \delta \) is path-connected. By Runge, there is a polynomial \( p_n \) such that

\[ \sup_{z \in \mathbb{D}} |p_n(z) - f_n(z)| \leq \frac{\varepsilon}{n}. \]

Finally, consider the polynomial

\[ q_n(z) = z p_n(z). \]

Then \( q_n(z) = 0 \) for all \( n \). If \( z \neq 0 \), then \( z \in \mathbb{C} \) for all \( n \) sufficiently large. Hence

\[ \lim_{n \to \infty} q_n(z) = \lim_{n \to \infty} z f_n(z) = r^{\frac{1}{n}} e^{i\frac{\Theta}{n}} \text{ for } z = r e^{i\Theta}, \text{ where } 0 < \Theta \leq 2\pi. \]

This is discontinuous on the positive real line.

\[ \square \]

Chapter 3: Approximation by Rationals

Contents

i) \( e, \pi \) irrational

ii) Continued Fractions (hopefuilly including \( e \))

iii) Transcendental Numbers (maybe proving that \( e, \pi \) are transcendental)

Theorem

\( e \) is irrational.

Proof

\[ e = 1 + \frac{1}{1} + \frac{1}{2!} + \frac{1}{3!} + \ldots \]

If \( e = \frac{p}{q} \), consider \( q! e \in \mathbb{Q} \).
But \( q! \varepsilon = q! \varepsilon + \frac{q!}{2!} + \ldots \frac{q!}{q^q} \) + \ldots 

Hence \( \frac{1}{q+1} + \frac{1}{(q+1)(q+2)} + \ldots \in \mathbb{N} \).

But \( 0 < \frac{1}{q+1} + \frac{1}{(q+1)(q+2)} + \ldots < \frac{1}{q+1} + \frac{1}{(q+1)^2} + \ldots = \frac{1}{2} < 1 \)

But there are no integers between 0 and 1.

\[
\text{Theorem}
\]

\( \pi \) is irrational.

\[
\text{Proof}
\]

Suppose that \( \pi = \frac{p}{q} \). Consider the polynomial \( f_n(x) = \frac{x^n}{n!} \).

\[
\text{Sketch Proof (to be expanded next time)}
\]

Consider \( \int_0^{\infty} f_n(x) \sin x \, dx \). Integrate by parts, repeatedly, until we get to \( f_n^{(2n+1)} = 0 \). Conclude that \( \int_0^{\infty} f_n(x) \sin x \, dx \in \mathbb{Z} \).

But \( 0 < \int_0^{\infty} f_n(x) \sin x \, dx < \frac{q^n (\frac{\pi}{2})^{2n}}{n!} \pi \to 0 \) as \( n \to \infty \).
Theorem

$\pi$ is irrational.

Proof

Suppose that $\pi = \frac{p}{q}$, and for $n \in \mathbb{N}$, consider $f_n(x) = \frac{q^n x^n (\pi - x)}{n!}$.

Observe that $f_n^{(m)}(0) = f_n^{(m)}(\pi) = 0$ for $m < n$, since $f_n$ vanishes to order $n$ at $0, \pi$. Also, $f_n^{(m)}(0) = f_n^{(m)}(\pi) = 0$ for $m > 2n$ (since $f_n$ is a polynomial).

Furthermore, $f_n^{(m)}(0), f_n^{(m)}(\pi) \in \mathbb{Z}$ for $m \geq n$, because by differentiating $x^n$ $n$ times, we get $r(r-1)\ldots(r-n+1)x^{r-n}$ or 0. If $r \geq n$, then

$$n! \mid r(r-1)\ldots(r-n+1) \quad \text{(since } \binom{r}{n} = \frac{r(r-1)\ldots(r-n+1)}{n!})$$

and the denominators of $q^n$ coming from $\pi$ are cancelled by the factor of $q^n$.

Now look at $\int_0^\pi f_n(x) \sin x \, dx$. One may evaluate this using many integrations by parts (keep differentiating $f_n$ and integrating $\sin x$). This will terminate after $2n+1$ steps, since $f_n^{(2n+1)}(\pi) = 0$.

Along the way, we will be evaluating expressions like

$$\left[ \pm f_n^{(m)}(x) \left[ \cos x \right]_0^\pi \right]_0^{\pi}.$$ All of these are integers, since $f_n^{(m)}(0), f_n^{(m)}(\pi) \in \mathbb{Z}$.

Hence, $\int_0^\pi f_n(x) \sin x \, dx \in \mathbb{Z}$. However, $\int_0^\pi f_n(x) \sin x \, dx < C$.

Also, since $x(\pi - x) \leq \left( \frac{\pi}{2} \right)^2$, $\forall x \in [0, \pi]$, we have $f_n(x) \leq \frac{C^2}{n!}$ where $C = q \left( \frac{\pi}{2} \right)^2$, which tends to 0 as $n \to \infty$.

Thus, if $n$ is large, $\int_0^\pi f_n(x) \sin x \, dx < 1$.
Continued Fractions

A continued fraction is an expression of the form
\[ a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \ldots}} , \quad a_0, a_1, a_2, \ldots \in \mathbb{N}. \]

i) Does this expression make sense / converge?

ii) Can we expand any real number like this?

iii) Are there some numbers whose continued fraction expansion is nice?

This is intimately related to the issue of approximating real numbers by rationals.

Notation: \( [a_0, a_1, a_2, \ldots] = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \ldots}} \).

We will look at finite truncations \( [a_0, \ldots, a_k] \).

Example

\[ [1, 1, 1, 1] = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1}}} = \frac{3}{2}, \quad [1, 1, 1, 1, 1] = \frac{8}{5}, \quad \text{then} \quad \frac{13}{8}, \frac{21}{13}, \text{etc.} \]

This appears to be related to the sequence of Fibonacci numbers.

Thus if \( \frac{p_k}{q_k} = [a_0, a_1, \ldots, a_k] \), then, when \( a_i = 1 \) & \( i \),

we appear to have \( p_k = p_{k-1} + p_{k-2} \), \( q_k = q_{k-1} + q_{k-2} \).

Proposition

Suppose that \( a_0, a_1, a_2, \ldots \in \mathbb{N} \). Write \( \frac{p_k}{q_k} = [a_0, a_1, \ldots, a_k] \)

in lowest terms. Then \( p_k = a_k p_{k-1} + p_{k-2} \), \( q_k = a_k q_{k-1} + q_{k-2} \).

Proof

Observe that if \( \frac{p'}{q'} = a_i + \frac{1}{\frac{p}{q}} \) (the continued fraction can

be evaluated by \( k \) operations of this type), then

\[ \frac{p'}{q'} = a_i + \frac{p}{q} = \frac{a_ip + q}{p}. \]

Hence

\[ \left( \frac{p}{q} \right) = \left( \begin{array}{c} a_1 \end{array} \right) \left( \begin{array}{c} a_i \end{array} \right) \left( \begin{array}{c} q \\ p \end{array} \right). \]

Note that if \( p, q \) have no common factor neither do \( p', q' \).
We start with $p_i = \frac{a_i}{i}$, already in lowest terms.

It follows that $\frac{p_k}{q_k} = [a_0, a_1, \ldots, a_k]$ is given by

$(p_k) = \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \ldots \begin{pmatrix} a_k & 1 \\ 1 & 0 \end{pmatrix} (1)

Write $M$ instead for $\begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \ldots \begin{pmatrix} a_k-2 & 1 \\ 1 & 0 \end{pmatrix}$

$(p_k) = M \begin{pmatrix} a_k-1 & 1 \\ 1 & 0 \end{pmatrix} (0)$

$(q_k) = M \begin{pmatrix} a_k-1 & 1 \\ 1 & 0 \end{pmatrix} (0) = M \begin{pmatrix} a_k-1 & a_k+1 \\ 0 & 0 \end{pmatrix}$

Hence $(p_k) = a_k (p_{k-1}) + (p_{k-2})$}

**Note that**

$(p_k, p_{k+1}) = \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \ldots \begin{pmatrix} a_k & 1 \\ 1 & 0 \end{pmatrix} (1)$ (follows as before)

Taking determinants, $p_k q_{k+1} - p_{k+1} q_k = (-1)^{k+1}$

This implies that $\frac{p_k}{q_k} - \frac{p_{k+1}}{q_{k+1}} = \frac{(-1)^{k+1}}{q_k q_{k+1}} \quad (\ast)$

**Note that from the recurrence relation $q_k = a_k q_{k-1} + q_{k-2}$, we have $q_k \to \infty$ as $k \to \infty$ (in fact, $q_k$ grows at least as fast as Fibonacci, i.e., exponentially). Thus $\left| \frac{p_k}{q_k} - \frac{p_{k+1}}{q_{k+1}} \right| \leq \frac{1}{k^2}$.**

Since $\sum \frac{1}{k^2}$ converges, $\frac{p_k}{q_k}$ is a Cauchy sequence, and hence converges. We have shown that if $a_0, a_1, a_2, \ldots \in \mathbb{N}$, then $\lim_{n \to 0} [a_0, \ldots, a_n] exists. We denote this by $[a_0, a_1, a_2, \ldots]$ or $\frac{1}{a_0 + a_1 + a_2 + \cdots}$

**Fact.**

$[1, 1, 1, \ldots] = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \cdots}}} = \frac{1 + \sqrt{5}}{2}$
\textbf{Proof:} \textit{k terms}

\[ [1, 1, \ldots, 1]_k = 1 + \frac{1}{[1, 1, \ldots, 1]_{k-1}}. \]

If \( \alpha = \lim_{k \to \infty} [1, 1, \ldots, 1]_k \), then \( \alpha = 1 + \frac{1}{\alpha} \). Solving the quadratic equation gives the result since \( \alpha = [1, 1, \ldots] \).

Also, for example, \( [1, 2, 2, \ldots] = \sqrt{2} \) and many similar expressions.

(*) immediately tells us that \( \frac{P_0}{q_0} < \frac{P_1}{q_1}, \frac{P_1}{q_2} > \frac{P_2}{q_2}, \frac{P_2}{q_3} < \frac{P_3}{q_3} \) etc.

Also, \( \frac{P_k}{q_k} - \frac{P_{k+2}}{q_{k+2}} = \frac{(-1)^k}{q_{k+1} q_{k+2}} + \frac{(-1)^{k+2}}{q_{k+2} q_{k+3}} = \frac{(-1)^k}{q_{k+1} q_{k+2}} (\frac{1}{q_{k+1}} - \frac{1}{q_{k+2}}) \)

So \( \frac{P_0}{q_0} < \frac{P_2}{q_2} < \frac{P_4}{q_4} < \ldots \) and \( \frac{P_1}{q_1} > \frac{P_3}{q_3} > \ldots \)
Continued Fractions

\[ P_k = a_k P_{k-1} + P_{k-2} \quad Q_k = a_k Q_{k-1} + Q_{k-2} \]

\[ \frac{P_k}{Q_k} = a_k + \frac{1}{\frac{P_{k+1}}{Q_{k+1}}} = \left( \frac{-1}{a_k + \frac{1}{a_{k+1} + \frac{1}{\ddots + \frac{1}{Q_k + \frac{1}{Q_{k+1}}}}} \right) \]

Representing a Real Number \((a \in (0, \infty))\)

Observe that for any \(k\), one can write \(x = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \ldots + \frac{1}{a_k + E_k}}} \) where \(0 \leq E_k < 1\). Indeed, to find the sequence of \(a_i\), repeat the following two operations:

- Take integer part, e.g., \(a_0 = \lfloor x \rfloor\). Subtract this.
- Reciprocate \((x \mapsto \frac{1}{x})\)

Fractional Part \(\frac{1}{E_k} - \left\lfloor \frac{1}{E_k} \right\rfloor\).

Note that \(E_k = \frac{1}{a_{k+1} + E_{k+1}}\), hence \(E_{k+1} = \left\lfloor \frac{1}{E_k} \right\rfloor\).

The map \(G: [0, 1) \to [0, 1)\), \(x \mapsto \left\lfloor \frac{x}{E_k} \right\rfloor\) is called the Gauss Map.

Write \(P_{\frac{x}{E_k}} = [a_0, \ldots, a_k, \bar{a}_k]\), \(\frac{\tilde{P}_k}{\tilde{Q}_k} = [a_0, \ldots, a_k + E_k] = x\)

We saw last time that
\[
\begin{pmatrix} P_k \\ Q_k \end{pmatrix} = \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \ldots \begin{pmatrix} a_{k-1} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}
\]

Similarly, \(\begin{pmatrix} \tilde{P}_k \\ \tilde{Q}_k \end{pmatrix} = \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \ldots \begin{pmatrix} a_{k-1} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_k + E_k \\ 1 \end{pmatrix}\)

Hence \(\begin{pmatrix} P_k & \tilde{P}_k \\ Q_k & \tilde{Q}_k \end{pmatrix} = \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \ldots \begin{pmatrix} a_{k-1} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_k & a_k + E_k \\ 1 & 1 \end{pmatrix}\)

Taking determinants gives \(|P_k \tilde{Q}_k - \tilde{P}_k Q_k| \leq 1\)

Hence \(|x - \frac{P_k}{Q_k}| = |\frac{\tilde{P}_k}{\tilde{Q}_k} - \frac{P_k}{Q_k}| \leq \frac{1}{2E_k Q_k} \to 0\)

(Use the recurrence relations for \(P_k, Q_k\) to see that they \(\to \infty\)).

Conclusion

Every positive real number has a continued fraction expansion, which is unique (exercise).
Further Properties of the Convergents

From the fact that \( \frac{p_0}{q_0} < \frac{p_2}{q_2} < \ldots < \alpha < \ldots < \frac{p_3}{q_3} < \frac{p_1}{q_1} \)
and the fact that \( \frac{p_k}{q_k} - \frac{p_{k+1}}{q_{k+1}} = \frac{1}{q_kq_{k+1}} \) we see that
\[
|\alpha - \frac{p_k}{q_k}| \leq \frac{1}{q_kq_{k+1}}. \quad \text{In particular, since } q_{k+1} > q_k, \text{ we have}
\]
\[
|\alpha - \frac{p_k}{q_k}| \leq \frac{1}{q_k^2} \quad \text{(no convergents tend to } \alpha \text{ rapidly)}
\]

Theorem (Best Rational Approximation)

Suppose that \( q \leq q_k \). Then \( |\alpha - \frac{p}{q}| \geq \frac{1}{q_kq_{k+1}} \), with equality \( q = q_k, \ p = p_k \) or \( q = q_{k-1}, \ p = p_{k-1} \).

Proof

We have \( |\alpha - \frac{p_k}{q_k}| \leq \frac{1}{q_kq_{k+1}}. \) However, if \( \frac{p}{q} \neq \frac{p_k}{q_k} \)
(WLOG, \( \frac{p}{q} \) is in lowest terms), then
\[
|\frac{p}{q} - \frac{p_k}{q_k}| = |\frac{p_k - p}{q_kq} - \frac{p_k - p}{q_kq_{k+1}}| \geq \frac{1}{q_kq} > \frac{1}{q_kq_{k+1}}. \quad \text{Hence}
\]
\[
\frac{p}{q}, \text{ if it is closer to } \alpha \text{ than } \frac{p_k}{q_k}, \text{ is on the other side of } \alpha \text{ from } \frac{p_k}{q_k}.
\]

Further work. We could have \( \frac{p_{k-1}}{q_{k-1}} = \frac{p_k}{q_k} \), in which case we are done. If not, \( |\frac{p}{q} - \frac{p_{k-1}}{q_{k-1}}| = \frac{p_{k-1} - p}{q_kq_{k-1}} \geq \frac{1}{q_kq_{k-1}}. \quad \text{Hence}
\]
\[
|\frac{1}{q} - \frac{p_{k-1}}{q_{k-1}}| \geq \left| \alpha - \frac{p_k}{q_k} \right|, \quad \text{so } \frac{p}{q} \text{ is equal to } \alpha \text{ or is not between } \frac{p_{k-1}}{q_{k-1}} \text{ and } \alpha. \quad \text{This covers all cases.}
\]

We have seen that convergents \( \frac{p_k}{q_k} \) are good approximations to \( \alpha \) in the sense that \( |\alpha - \frac{p_k}{q_k}| \leq \frac{1}{q_k^2} \).

Remarkably, the converse is true up to a factor of 2.

Theorem

Let \( \alpha \in (0, \infty), \ |\alpha - \frac{p}{q}| \leq \frac{1}{2q^2}, \ \frac{p}{q} \text{ in lowest terms.} \)

Then \( \frac{p}{q} \) is one of the convergents to the continued fraction of \( \alpha \).
Proof

Since the sequence of denominators $q_k$ is increasing, we choose $k$ such that $q_k \leq q < q_{k+1}$.

If $P_{q_k} = \frac{p_k}{q_k}$ we are done, so suppose not. Then

$$|P_{q_k} - \frac{p_k}{q_k}| \geq \frac{1}{q_{k+1}q_k} > \frac{1}{q_k q_{k+1}} \Rightarrow |\alpha - \frac{p_k}{q_k}|$$

Case 1

$P_{q_k} = \frac{p_k}{q_k}$. Then $P_{q_k}$ cannot lie between $\frac{p_k}{q_k}$ and $\alpha$.

Hence we may assume (the other case being similar) that $\frac{p_k}{q_k} < \alpha < \frac{p_{k+1}}{q_{k+1}}$. By the fact that $\frac{p_{k+1}}{q_{k+1}}$ is a better approximation to $\alpha$ than $p_k/q_k$, this implies that $\frac{p_{k+1}}{q_{k+1}} < \alpha < \frac{p_{k+1}}{q_{k+1}} < P_{q_k}$.

Consider first of all the case $q > \frac{1}{2} q_{k+1}$. Then

$$|P_{q_k} - \alpha| \geq |P_{q_{k+1}} - P_{q_k}| = \frac{1}{q_{k+1}q_k} \geq \frac{1}{2q^2} \times$$

Otherwise, suppose that $q < \frac{1}{2} q_{k+1}$. Then

$$|P_{q_k} - \alpha| = |P_{q_k} - \frac{p_k}{q_k} - 1| = |\frac{p_k}{q_k} - \alpha| \geq \frac{1}{q_{k+1}q_k} - \frac{1}{q_k q_{k+1}}$$

$$= \frac{1}{q_k q_k} (\frac{1}{q_k} - \frac{1}{q_{k+1}}) > \frac{1}{2q^2} \times$$
Two slightly tricky statements about continued fractions

Let $\alpha$ be a real number, and let $[a_0, a_1, a_2, \ldots]$ be its continued fraction expansion. Write $\frac{p_k}{q_k}$ for the convergents. To avoid annoyances, I'll assume that this expansion is infinite. This is so if and only if $\alpha$ is irrational. The less obvious direction is the only if direction, which is the statement that the continued fraction expansion of a rational number is finite. To see this, suppose that $\alpha = \frac{p}{q}$ is rational and has an infinite continued fraction expansion. Certainly, for each $k$, we have $\frac{p}{q} \neq \frac{p_k}{q_k}$, and so

$$|\alpha - \frac{p_k}{q_k}| = \left| \frac{p}{q} - \frac{p_k}{q_k} \right| \geq \frac{1}{qq_k},$$

However we also know that $|\alpha - \frac{p_k}{q_k}| \leq \frac{1}{q_k^2}$. Hence $q_k \leq q$ for all $k$, a contradiction.

We proved in lectures that

$$\frac{p_0}{q_0} < \frac{p_2}{q_2} < \ldots < \alpha < \ldots < \frac{p_3}{q_3} < \frac{p_1}{q_1}.$$

Also, remember, we have

$$|\alpha - \frac{p_k}{q_k}| < \frac{1}{q_k q_{k+1}}$$

for all $k$.

We will make very frequent use of the following observation: if $\frac{a}{b}$ and $\frac{a'}{b'}$ are distinct fractions in lowest terms, then

$$|\frac{a}{b} - \frac{a'}{b'}| = \frac{|ab' - a'b|}{bb'} \geq \frac{1}{bb'}.$$

Now down to business. We first prove that convergents to $\alpha$ are record approximants in a certain sense.

**Theorem 1.** Let $\alpha$ be an irrational real number and let $\frac{p_k}{q_k}$ be the convergents to $\alpha$. Suppose that $k$ is odd. Then any rational in the interval $(\frac{p_{k-1}}{q_{k-1}}, \frac{p_k}{q_k})$ has denominator greater than $q_k$. A similar statement holds when $k$ is even.

**Proof.** Suppose that some rational $\frac{p}{q}$ in lowest terms lies in the interval stated and that $q \leq q_k$. We split into two cases.

**Case 1.** $\alpha < \frac{p}{q} < \frac{p_k}{q_k}$. Then

$$|\frac{p_k}{q_k} - \frac{p}{q}| \geq \frac{1}{qq_k} > \frac{1}{q_k q_{k+1}} > |\alpha - \frac{p_k}{q_k}|,$$

contradiction.

**Case 2.** $\frac{p_{k-1}}{q_{k-1}} < \frac{p}{q} < \alpha$. Then

$$|\frac{p_{k-1}}{q_{k-1}} - \frac{p}{q}| \geq \frac{1}{qq_{k-1}} > \frac{1}{q_k q_{k-1}} > |\alpha - \frac{p_{k-1}}{q_{k-1}}|,$$

contradiction. \qed
**Lemma 1.** For all \( k \) we have \( |\alpha - \frac{p_{k-1}}{q_{k-1}}| > |\alpha - \frac{p_k}{q_k}|. \)

**Proof.** Since \( \frac{p_{k-1}}{q_{k-1}} \) and \( \frac{p_k}{q_k} \) lie on opposite sides of \( \alpha \), it is enough to show that \( |\frac{p_{k-1}}{q_{k-1}} - \frac{p_k}{q_k}| > 2|\alpha - \frac{p_k}{q_k}| \). However we have \( |\frac{p_{k-1}}{q_{k-1}} - \frac{p_k}{q_k}| \geq \frac{1}{q_{k-1}q_k} \) and \( |\alpha - \frac{p_k}{q_k}| < \frac{1}{q_kq_{k+1}} \), and so it suffices to show that \( q_{k+1} \geq 2q_{k-1} \). This follows from the fact that \( q_{k+1} = q_k q_{k+1} \). \( \square \)

Putting these facts together, we immediately obtain the following result.

**Theorem 2.** Suppose that \( \alpha \) is an irrational number and that \( \frac{p_k}{q_k} \) is a convergent to \( \alpha \). Then the only rational \( \frac{p}{q} \) with \( q \leq q_k \) for which \( |\alpha - \frac{p}{q}| \leq |\alpha - \frac{p_k}{q_k}| \) is \( \frac{p_k}{q_k} \) itself.

Secondly, we prove that all good approximants to \( \alpha \) are convergents.

**Theorem 3.** Suppose that \( \frac{p}{q} \) is a fraction in lowest terms and that \( |\alpha - \frac{p}{q}| \leq \frac{1}{2q^2} \). Then
\[
\frac{p}{q} = \frac{p_k}{q_k} \text{ for some } k.
\]

**Proof.** Since the denominators \( q_k \) are increasing, we may select a unique \( k \) such that \( q_k \leq q < q_{k+1} \). Suppose that \( \frac{p_k}{q_k} < \alpha < \frac{p_{k+1}}{q_{k+1}} \) (the other case is very similar).

First note that if \( \frac{p}{q} < \frac{p_k}{q_k} \) then
\[
|\alpha - \frac{p}{q}| > |\frac{p}{q} - \frac{p_k}{q_k}| \geq \frac{1}{qq_k} \geq \frac{1}{q^2},
\]
contrary to assumption.

If \( \frac{p}{q} = \frac{p_k}{q_k} \) then we are done. Suppose, then, that \( \frac{p_k}{q_k} < \frac{p}{q} \). We have
\[
|\frac{p}{q} - \frac{p_k}{q_k}| \geq \frac{1}{qq_k} > \frac{1}{q_kq_{k+1}} \geq |\alpha - \frac{p_k}{q_k}|,
\]
so in fact \( \frac{p_k}{q_k} < \alpha < \frac{p}{q} \). By the previous theorem, we cannot have \( \frac{p}{q} < \frac{p_{k+1}}{q_{k+1}} \), and hence \( \frac{p_k}{q_k} < \alpha < \frac{p_{k+1}}{q_{k+1}} < \frac{p}{q} \). We now divide into two cases.

*Case 1. (q large).* Suppose that \( q \geq \frac{1}{2} q_{k+1} \). Then
\[
|\alpha - \frac{p}{q}| > |\frac{p}{q} - \frac{p_{k+1}}{q_{k+1}}| \geq \frac{1}{q_{k+1}} \geq \frac{1}{2q^2},
\]
contrary to assumption.

*Case 2. (q small).* Suppose that \( q < \frac{1}{2} q_{k+1} \). Then
\[
|\alpha - \frac{p}{q}| = |\frac{p}{q} - \frac{p_k}{q_k}| - |\alpha - \frac{p_k}{q_k}| \geq \frac{1}{qq_k} - \frac{1}{q_kq_{k+1}} = \frac{1}{q_k} \left( \frac{1}{q} - \frac{1}{q_{k+1}} \right) > \frac{1}{2q^2},
\]
also a contradiction. \( \square \)
Continued Fractions

Periodic Continued Fractions and Quadratic Irrationals

We saw that \( \frac{1 + \sqrt{15}}{2} = [1, 1, \ldots] \) is periodic. Define a quadratic irrational to be an \( \alpha \) which is not rational, but satisfies some quadratic equation \( a\alpha^2 + b\alpha + c = 0 \), \( a, b, c \in \mathbb{Z} \). These are numbers of the form \( \frac{A + B\sqrt{D}}{C} \), \( A, B, C, D \in \mathbb{Z} \), \( D > 0 \).

Theorem

A number \( \alpha \) has periodic continued fraction expansion (that is, the partial quotients \( a_i \) repeat from some point on).

\[ \alpha \text{ is a quadratic irrational.} \]

Proof

We will first show the easier direction, namely that periodic continued fractions represent quadratic irrationals. We begin with the purely periodic case \( \alpha = [a_0, a_1, \ldots, a_k, a_0, a_1, \ldots] \).

Then \( \alpha \) solves the equation

\[ \alpha = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\ddots}}} \]

Rearranging, (subtract \( a_0 \), reciprocate, subtract \( a_1 \), etc.) gives an equation of the form

\[ \frac{c_1\alpha + c_2}{c_3\alpha + c_4} = \alpha, \quad C_i \in \mathbb{Z} \]

This is a quadratic equation for \( \alpha \). Since \( \alpha \) is uniquely determined by \( a_0, a_1, \ldots \), this equation cannot be trivial, and so \( \alpha \) is a quadratic irrational.
This can be used in practice to evaluate any repeated continued fraction.

For the general case, suppose that \( \alpha = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \cdots}} \) with \( \beta \) purely periodic. Now observe that \( \beta \) is a quadratic irrational, and the set of quadratic irrationals is closed under reciprocation and addition of integers.

\[
e.g. \quad \frac{C}{A + BL} = \frac{C(A - BL)}{A^2 - 2BL}
\]

The other direction is much trickier.

Suppose that \( \alpha \) satisfies \( a_0 x^2 + b_0 x + c = 0 \), which we will write as \( (x - 1)(a_0 \ b_0 \ c) (x) = 0 \). Now we define \( r_n \) by \( \alpha = [a_0, a_1, a_2, \ldots, a_{n-1}, r_n] \).

Then, expanding out \([a_0, a_1, a_2, \ldots, a_{n-1}, r_n] \), we get \( \alpha = \frac{P_n}{Q_n} = \frac{P_n \cdot r_{n-1} + P_{n-2}}{Q_n \cdot r_{n-1} + Q_{n-2}} \) from the recurrence relations for convergents.

Hence \( (\alpha) \times (\frac{P_{n-1}}{Q_{n-1}}) \) (we write \( \alpha \times \frac{P_{n-1}}{Q_{n-1}} \)).

Hence \( (r_n 1)(\frac{P_{n-1}}{Q_{n-1}}) (a_0 \ b_0 \ c) (\frac{P_{n-1}}{Q_{n-1}} \cdot r_{n-2}) = 0 \).

Hence \( r_n \) satisfies the quadratic equation

\[A_n r_n^2 + B_n r_n + C_n = 0\]

where \( A_n = a_0 P_{n-1} + b_0 P_{n-2} + c_0 Q_{n-1} \)

\( B_n = a_0 P_{n-2} + b_0 P_{n-3} + c_0 Q_{n-2} \)

\( C_n = a_0 + b_0 r_{n-1} + c_0 r_{n-2} \).

Taking determinants, \( \left| \begin{array}{cc} A_n & B_n \frac{1}{Q_n} \\ B_n & C_n \end{array} \right| = \left| \begin{array}{cc} P_{n-1} & P_{n-2} \\ Q_{n-1} & Q_{n-2} \end{array} \right| \frac{a_0}{\frac{b_0}{Q_n}} \frac{1}{C_n} \)

\( = \left| \begin{array}{cc} a_0 & b_0 \frac{1}{Q_n} \\ b_0 & c_0 \end{array} \right| \), since \( P_{n-1} Q_n - P_{n-2} Q_{n-1} = (1)^n \)
Cleariy $A_n, B_n, C_n \in \mathbb{Z}$. We claim that $|A_n|, |C_n|$ are bounded independently of $\|n\|$. It then follows that $B_n$ is too (since $B_n^2 - 4A_nC_n = b^2 + 4ac$). Hence $r_n$ satisfies one of boundedly many equations. Hence $r_n = r_{n'}$ for some $n \neq n'$, at which point we are done.

We will show that $A_n$ is bounded ($C_n$ very similar),

$$A_n = q_{n-1}^{-2} \left( a \left( \frac{p_n}{q_{n-1}} \right)^2 + b \left( \frac{p_n}{q_{n-1}} \right) + c \right) = a q_{n-1}^{-2} \left( \frac{p_n}{q_{n-1}} - \alpha \right) \left( \frac{p_n}{q_{n-1}} - \bar{\alpha} \right),$$

where $\alpha, \bar{\alpha}$ are the roots of $ax^2 + bx + c = 0$.

But $|\frac{p_n}{q_{n-1}} - \alpha| \leq \frac{1}{q_{n-1}^2}$ since $\frac{p_n}{q_{n-1}}$ is a convergent.

Also, $|\frac{p_n}{q_{n-1}} - \bar{\alpha}| \leq |\alpha| + 1 + |\bar{\alpha}|$ if $n$ is large enough that $|\frac{p_n}{q_{n-1}} - \alpha| \leq 1$.

Putting this together gives $|A_n| \leq \alpha \left( |\alpha| + 1 + |\bar{\alpha}| \right)$.

* The Gauss Map and Continued Fraction Expansions of Typical $\alpha$ Remember that $\alpha = [a_0, a_1, \ldots, a_{k-1}, a_k + E_k]$ and we have $E_k = \frac{1}{a_{k+1} + E_{k+1}}$, with $E_{k+1} = T(E_k)$, where $T(x) = \left\lfloor \frac{1}{x} \right\rfloor$ is the Gauss Map.

Question

What is the 'probability' that $E_{k+1} \in [0, c]$ for some $c$?

One might expect

i) The probability that $E_k \in [0, c]$ is the same as for $E_{k+1}$

ii) The probability is given by a formula

$$P(E_k \in [0, c]) = \int_0^c f(x) \, dx$$
Recall \( \alpha = [a_0, a_1, ..., a_{k-1}, a_k, E_k] \), \( T(E_k) = E_{k+1} \)

\[ T(x) = \{ \frac{1}{2} \} \]. For random \( \alpha \in [0, 1] \), one imagines that
\( P(E_k \in [0, c]) \) should be independent of \( k \) and should be
"nice", \( P(E_k \in [0, c]) = \int_0^c f(x) \, dx \).
This is a reasonable thing to imagine, akin to decimal expansion, where

\( P(k\text{th digit} = m) = \frac{1}{10} \).

The analog of \( E_k \) is the base 10 expansion after the \( k \)th digit, 0.00... 0 \( d_k d_{k+1} \ldots \). The remaining
\( 0.\ldots d_k d_{k+1} \ldots \) is uniformly distributed in \([0, 1] \).

What is \( f \)?

We should have \( P(E_{k+1} \in [0, c]) = \int_0^c f(x) \, dx \)
but also \( P(E_k \in \bigcup_{j=\frac{k}{3+2c}}^{\frac{k+1}{3+2c}}) = \sum_{j=\frac{k}{3+2c}}^{\frac{k+1}{3+2c}} f(x) \, dx \).

Since \( T^{-1}([0, c]) = \bigcup_{j=\frac{k}{3+2c}}^{\frac{k+1}{3+2c}} \)

It has been observed that \( f(x) = \frac{1}{1+x^2} \) satisfies this

\[ \int_0^c f(x) \, dx = \log (1+c) \]

\[ \sum_{j=\frac{k}{3+2c}}^{\frac{k+1}{3+2c}} f(x) \, dx = \sum_{j=\frac{k}{3+2c}}^{\frac{k+1}{3+2c}} \left( \log (1+j) - \log (1+\frac{j}{3+2c}) \right) \]

\[ = \lim_{m \to \infty} \left( \sum_{j=\frac{k}{3+2c}}^{\frac{k+1}{3+2c}} \left( \log (mt) - \log (1+mt+c) \right) \right) + \log (1+c) \]

But \( \lim_{m \to \infty} (\log (m+1) - \log (1+m+c)) = 0 \)

To make sure that \( \int_0^1 f(x) \, dx = 1 \), we take

\[ f(x) = \frac{1}{\log 2} \frac{1}{1+x} \].

The measure defined by

\[ m(A) = \int_A f(x) \, dx \] is called the Gauss Measure.

This "should be" the probability that \( E_k \) lies in \( A \).
** To put this on a rigorous footing, proceed as follows. The Gauss measure is invariant for the Gauss Map, $\mu(T^{-1}(A)) = \mu(A)$. We proved this when $A = [0, c]$ and the general case follows by taking limits (i.e., when $A$ is Lebesgue measurable). Key fact: $T$ is ergodic for this measure.

Ergodic means that $T$ has no non-trivial invariant sets, i.e. $T^{-1}(A) = A \Rightarrow \mu(A) = 0$ or 1.

Ergodic Theorem

"Time Averages = Space Averages"

$$\lim_{N \to \infty} \frac{1}{N} (\psi(x) + \psi(T(x)) + \ldots + \psi(T^{N-1}(x)) = \frac{1}{\log 2} \int_{\mathbb{R}} \psi(x) \frac{dx}{1 + e^x}$$

for almost all $x$ and "nice" $\psi$.

In other words, for almost all $x$, $\{x, T(x), T^2(x), \ldots\}$ are distributed according to the Gauss measure.

What is the probability that the $k^{th}$ partial quotient of a random $x$ is equal to $m$?

This is $P(E_k \in \left(\frac{1}{m+1}, \frac{1}{m}\right]) = \frac{1}{\log 2} \int_{\frac{1}{m+1}}^{\frac{1}{m}} \frac{dx}{1 + e^x}$

$$= \frac{1}{\log 2} \left( \log (1 + \frac{1}{m}) - \log (1 + \frac{1}{m+1}) \right) = \frac{\log 2}{\log (1 + \frac{1}{m+1})} \approx \frac{\log 2}{m}$$

(Hence the fact that $292$ is a partial quotient of $\pi$, meaning that $\frac{355}{113}$ is such a good approximation to $\pi$, has "probability about 1%")

* Continued Fraction of $e$

$$e = [2, 1, 2, 1, 4, 1, 1, 6, 1, 1, 8, \ldots ]$$

$$= [1, 0, 1, 1, 2, 1, 1, 4, \ldots ]$$
Here is a sketch proof:

If \( p_n/q_n \) are the partial convergents, since \( a_{3n} = 1 \), \( q_{3n+1} = 2n \), \( a_{3n+2} = 1 \), we should have

\[ P_n = P_{n-1} + P_{n-2}, \quad P_{3n} = 2nP_n + P_{3n-1}, \quad P_{3n+2} = P_{3n+1} + P_n \]

and similar for \( q_n \).

Now consider \( A_n = \int_0^1 t^n (1-t)^e dt \)
\( B_n = \int_0^1 t^n (1-t)^{e^2} dt \)
\( C_n = \int_0^1 t^n (1-t)^{e^3} dt \)

Claim

\[ p_{3n} - 2q_{3n} = A_n, \quad p_{3n+1} - 2q_{3n+1} = B_n, \quad p_{3n+2} - 2q_{3n+2} = C_n. \]

Proof by Induction

Given the recurrence relations, it is enough to check \( n = 0 \) (easy),
and: 1. \( A_n = C_{n-1} + B_{n-1} \), 2. \( B_n = 2n A_n + C_{n-1} \)

\( C_n = A_n + B_n \)

To prove 1 look at \( \int_0^1 \frac{d}{dt} \left( t^n (1-t)^e \right) dt = 0 \)
To prove 2 look at \( \int_0^1 \frac{d}{dt} \left( t^n (1-t)^{e^2} \right) dt = 0 \)

But \( |A_n|, |B_n|, |C_n| \leq \frac{C_n}{n!} \), so it follows that

\[ \frac{p_n}{q_{3n}}, \frac{p_{3n+1}}{q_{3n+1}}, \frac{p_{3n+2}}{q_{3n+2}} \to e. \]

Corollary

\( e^2 \) is irrational. Indeed, if \( e^2 \) were rational, the continued fraction expansion of \( e^2 \) would be periodic.

Open Problems

1. Does \( 3/2 \) have bounded partial quotients?
2. (Littlewood Conjecture) Suppose that \( \alpha, \beta \) are independent.
Are there arbitrarily large $q_n$ such that $q_n \left[ a_3 \right] \beta \geq 0$?

Note that if the $q_n$ are the denominators of convergents to $\alpha$, then $|1 - \frac{P_n}{Q_n}| \leq \frac{1}{Q_n^2} \Rightarrow q_n \left[ a_3 \right] \leq 1$, and similarly for $\beta$. But how to do this simultaneously?

**Transcendental Numbers**

**Definition**

$\alpha$ is algebraic if it satisfies a non-trivial polynomial equation

$$a_n \alpha^n + a_{n-1} \alpha^{n-1} + \ldots + a_1 \alpha + a_0 = 0,$$

where $a_i \in \mathbb{Z}$ for all $i$.

E.g., $\alpha = \sqrt{2}, \sqrt{3}, \sqrt{5} + \sqrt{7}, 10^{1/2}$ etc.

If $\alpha$ is not algebraic, it is transcendental. Cantor:

There exist transcendental numbers (Numbers and Sets).
Transcendental Numbers

The key idea is that if \( \alpha \) is algebraic then \( \alpha \) is either rational or is not "too well approximated" by rationals.

**Theorem (Liouville)**

Suppose that \( \alpha \) satisfies a polynomial \( a_n \alpha^n + \ldots + a_0 = 0 \) with \( a_i \in \mathbb{Z} \). Suppose that \( \alpha \) is not rational. Then there is a constant \( c(\alpha) \) such that \( |\alpha - \frac{p}{q}| > \frac{c(\alpha)}{q^n} \) for all integers \( p, q \), \( p, q \neq 0 \).

**Proof**

WLOG \( P_n \) is not a root of the equation satisfied by \( \alpha \) (if it is, divide by \( x - P_n \) and clear denominators).

Since an integer that is not zero must have magnitude at least 1, \( |a_n (\frac{p}{q})^n + \ldots + a_0| > \frac{1}{q^n} \) since the left-hand side is a non-zero integer over \( q^n \). Comparing with \( a_n \alpha^n + \ldots + a_0 = 0 \), we get \( |a_n (\alpha^n - (\frac{p}{q})^n) + \ldots + a_0| \geq \frac{1}{q^n} \)

\[ |a_n| |\alpha^n - (\frac{p}{q})^n| + \ldots + |a_0| \geq \frac{1}{q^n} \] \((*)\)

But \( |\alpha^n - (\frac{p}{q})^n| = |\alpha - \frac{p}{q}| |\alpha^{n-1} + \alpha^{n-2}(\frac{p}{q}) + \ldots + (\frac{p}{q})^{n-1}| \)

If \( |\alpha - \frac{p}{q}| > 1 \) then the theorem is immediate. Otherwise, the preceding is bounded by \( |\alpha - \frac{p}{q}| c(m, \alpha) \), where

\[ c(m, \alpha) = |\alpha|^{m-1} + |\alpha|^{m-2}(\alpha + 1) + \ldots + (\alpha + 1)^{m-1} \]

Substituting back into \((*)\), we get

\[ |\alpha - \frac{p}{q}| \hat{c}(\alpha) \geq \frac{1}{q^n} \]

where \( \hat{c}(\alpha) = |a_n| c(n, \alpha) + |a_{n-1}| c(n-1, \alpha) + \ldots + |a_0| \)
To make a transcendental number, one need only write down a non-rational that is very well approximated by rationals, such as \( \alpha = \sum_{j} 10^{-3j} \).

Theorem
\( \alpha = \sum_{j} 10^{-3j} \) is transcendental.

Proof
Let \( q = 10^{N+1} \). Then \( \alpha \) is equal to some rational \( \frac{p}{q} \), plus some error \( \frac{\sum_{j} 10^{-3j}}{2^{N+1}} \) which is at most \( 2 \cdot 10^{-(N+1)} \), which is \( 2q^{-N-1} \). For any fixed \( n, C \) this is smaller than \( \frac{p}{q} \) if \( N \), and hence \( q \) are sufficiently large. Hence \( \alpha \) is not the root of a polynomial of degree \( n \), for any \( n \).

Remark
We found a sufficient condition (being exceptionally well approximated by rationals) for being transcendental. It was not necessary. We will see in a moment that \( e \) is transcendental. However, \( e \) is not incredibly well approximated by rationals. In fact, \( |e - \frac{p}{q}| > \frac{c}{q^{2+\epsilon}} \) for every \( \epsilon > 0 \), which follows from the continued fraction expansion of \( e \).

2. Roth famously proved the following improvement of Liouville's result: If \( \alpha \) is algebraic and not rational, then for every \( \epsilon > 0 \), there is a constant \( C = C(\alpha, \epsilon) \) such that \( |\alpha - \frac{p}{q}| > \frac{C(\alpha, \epsilon)}{q^{2+\epsilon}} \). But unfortunately, \( C(\alpha, \epsilon) \) is not effectively computable.
Theorem

$e$ is transcendental. We will consider integrals of the form

$I(t) = \int_0^t e^{-u} f(u) \, du$ where $f$ is a polynomial.

Proof

Integrating by parts gives $I(t) = e^t f(0) - f(t) + \int_0^t e^{-u} f'(u) \, du$.

Doing this $D + 1$ times, where $D = \deg(f)$, gives

$I(t) = e^t \sum_{i=0}^D f^{(i)}(0) - \sum_{i=0}^D \frac{t^i}{i!} f^{(i)}(t)$ \hspace{1cm} (*)

Suppose for contradiction that $b_0 + b_1 e + \ldots + b_r e^r = 0$ with $b_0, b_1, \ldots, b_r \in \mathbb{Z}$, $b_r \neq 0$. Let $p$ be a large prime and consider $f(x) = x^{p-1}(x-1)^p \ldots (x-r)^p$ (the clever bit)

Applying (*) with this $f$, we get

$\sum_{i=0}^D b_i I(i) = -\sum_{i=0}^D b_i \frac{D^i}{i!} f^{(i)}(i)$ \hspace{1cm} ($D = \deg(f) = rp + r - 1$)

Key observations

1. The left hand side behaves like $C^p$ as $p \to \infty$

2. Everything on the right hand side is divisible by $p!$ except for one term only divisible by $(p-1)!$ \Rightarrow Right Hand Side $\geq (p-1)!$

Details

1. We have $|f(x)| \leq (r^{r+1})^p$ for $0 \leq x \leq r$ (very crude). Hence

$|I(t)| \leq r (e^r r^{r+1})^p$ for $0 \leq t \leq r$.

Thus, $\sum_{i=0}^D b_i I(i) \leq r (r+1) \max \{|b_i| (e^r r^{r+1})^p \leq (2e^r r^{r+1})^p$ if $p$ is large enough.

2. If we differentiate $f(x) = x^{p-1}(x-1)^p \ldots (x-r)^p$ less than $p$ times, then set $x = 0, 1, \ldots, r$, we will always
get 0 except that \( f^{(p-1)}(0) = (p-1)! \cdot (-1)^r \cdot r! \). (product rule)

On the other hand, if we differentiate \( p \) or more times, \( f^{(\delta)}(i) \) is always divisible by \( p! \). Hence

\[
\text{RHS} = -b \cdot (p-1)! \cdot (-1)^r \cdot r! + \text{multiple of } p!.
\]

Choose \( p \) so large that \( p \times b \) and \( p \times r! \). Then

\[
\text{RHS} \geq (p-1)!. \quad \text{But for } p \text{ large enough,}
\]

\[
(p-1)! > C^p, \quad \text{for any } p.
\]
The Baire Category Theorem

Let \( X \) be a metric space. We will be talking about complete metric spaces — every Cauchy sequence has a limit.

If \( A \subseteq X \), then we say that \( A \) is dense if \( A \) intersects every open ball \( B_{r}(x) \) in \( X \).

E.g., the rationals, \( \mathbb{Q} \), and irrationals, \( \mathbb{R} \setminus \mathbb{Q} \), are dense in \( \mathbb{R} \).

Theorem (Baire Category Theorem)

Let \( X \) be a non-empty complete metric space. Let \( (A_{n})_{n=1}^{\infty} \) be a sequence of dense open sets. Then \( \bigcap_{n=1}^{\infty} A_{n} \) is non-empty (and in fact dense).

Equivalent Formulation

Let \( X \) be a non-empty complete metric space. Suppose that \( X = \bigcup_{n=1}^{\infty} F_{n} \) with each \( F_{n} \) closed. Then one of these sets \( F_{n} \) has non-empty interior; that is to say that it contains some \( B_{\varepsilon}(x) \), \( \varepsilon > 0 \).

"A complete metric space is not a countable union of small closed sets."

Proof

We will repeatedly use the fact that \( B_{\varepsilon}(x) \supseteq \overline{B_{\varepsilon}(x)} \) since \( \overline{A} \) is dense.

Pick a ball \( B_{\varepsilon_{0}}(x_{0}) \) in \( X \). This intersects \( A_{1} \). Since \( A_{1} \) is open, \( A_{1} \cap B_{\varepsilon_{0}}(x_{0}) \) contains an open ball, and hence, contains \( \overline{B_{\varepsilon_{1}}(x_{1})} \) of some open ball. Since \( A_{2} \) is dense, it intersects \( B_{\varepsilon_{1}}(x_{1}) \), and the intersection contains an open ball, and hence the closure \( \overline{B_{\varepsilon_{2}}(x_{2})} \) of some open ball.
We continue in this way to produce a nested sequence $B_{E_i}(x_i)$ such that $B_{E_i}(x_i) \subseteq A_i, \ldots, A_i, B_{E_i}(x_i+i) \subseteq B_{E_i}(x_i) \cap A_i$. Make sure that the $E_i$ are chosen so that $E_i \to 0$. Then the centres of these balls, $x_i$, form a Cauchy sequence. Thus, there is some $x$ such that $x_i \to x$. We must have $x \in \overline{B_{E_i}(x_i)}$ and hence $x \in \bigcap_{n=1}^{\infty} A_n$. To see that $\bigcap_{n=1}^{\infty} A_n$ is in fact dense, replace $X$ by $X' = \overline{B_{E_i}(x)}$ for any $x \in X, \epsilon > 0$. $X'$ is a complete metric space, and the sets $A' = A \cap X'$ are open in this subspace topology on $X'$ and are dense. Thus $\bigcap A'$ is non-empty and lies in $\overline{B_{E_i}(x)}$.

Proposition

There exists a continuous, nowhere differentiable function $f : [0, 1] \to \mathbb{R}$ (Such a function can be constructed explicitly).

Proof

Let $X = C[0, 1]$ be the metric space of continuous functions on $[0, 1]$ together with the supremum norm. This is complete (uniform limits of continuous functions are continuous).

We will define a sequence $(A_n)_{n=1}^{\infty}$ of closed $A_n \subseteq X$ such that

i) If $f$ is differentiable at some point of $(0, 1)$, then $f$ lies in some $A_n$.

ii) Each $A_n$ has empty interior.
The Baire Category Theorem then implies that $X \not\in \bigcup_{n=1}^{\infty} A_n$. Then any $f \in X \setminus \bigcup_{n=1}^{\infty} A_n$ is continuous but nowhere differentiable. Define $A_n$ to be the set of $f \in X$ such that there exists $x \in [0, 1]$ with the property that $\left| \frac{f(x) - f(y)}{x - y} \right| \leq n$ whenever $0 < |x - y| \leq \frac{1}{n}$. Note that if $f$ is differentiable at $x$, then $\lim_{y \to x} \frac{f(y) - f(x)}{y - x} = f'(x)$ exists. Thus if $n$ is sufficiently large, $f \in A_n$. This is 1).

Why is $A_n$ closed?

Suppose that $f_i$ is a sequence of functions in $A_n$ and that $f_i \to f$. We need to show that $f \in A_n$. We know that for each $i$, there is an $x_i$ such that $\left| f_i(y) - f_i(x_i) \right| \leq n$ whenever $0 < |y - x_i| \leq \frac{1}{n}$. By Bolzano-Weierstrass, there is a subsequence of the $x_i$ converging to some point $x$. Relabelling, we assume that in fact, $x_i \to x$. Suppose that $0 < |y - x| < \frac{1}{n}$. Then if $i$ is large enough, we have $0 < |y - x_i| < \frac{1}{n}$ and so $\left| \frac{f_i(y) - f_i(x_i)}{y - x_i} \right| \leq n$. Now let $i \to \infty$. Since $f_i \to f$ uniformly and each $f_i$ is continuous, we get

\[
\lim_{i \to \infty} f_i(y) = f(y), \quad \lim_{i \to \infty} f_i(x_i) = f(x), \quad \text{and hence}
\]

\[
\left| \frac{f(y) - f(x)}{y - x} \right| \leq n.
\]

Since $f$ is continuous, the same bound holds when $|y - x| = \frac{1}{n}$ as well. Hence $A_n$ is closed.

To complete the proof, we need to show that $A_n$ has empty interior.
"Close to $f$ is any function with horibly unbounded slope."

We do this in two steps. Let $\varepsilon > 0$. Then

i) $f$ is within $\frac{\varepsilon}{2}$ (in the uniform norm) of a piecewise linear function (we can subdivide $[0, 1]$ into finitely many segments $[x_i, x_{i+1}]$ on which $g$ is linear).

ii) Within $\frac{\varepsilon}{2}$ of any piecewise linear function is a function not in $A_n$.

i), ii) combine to show that within $\varepsilon$ of $f$, i.e. in $B_\varepsilon(f)$, there is a continuous function not in $A_n$. Thus $A_n$ has empty interior.

Proof of i)

$f$ is uniformly continuous, hence if $\delta$ is small enough, then $|f(x) - f(y)| < \frac{\varepsilon}{4}$ whenever $|x-y| < \delta$. Partition $[0, 1]$ into finitely many intervals $[x_i, x_{i+1})$ of width $\leq \delta$. Define $g: [0, 1] \to \mathbb{R}$ by setting $g(x_i) = f(x_i)$ for all $i$ and taking $g$ linear on $[x_i, x_{i+1})$.

It follows that $\|f - g\| \leq \frac{\varepsilon}{2}$. 
Topics in Analysis

Baier Category Theorem and Pathological Functions

If every continuous function was differentiable at some point, then we would have \( X = \bigcup_{n=1}^{\infty} A_n \). Baier \( \Rightarrow \) some \( A_n \) has non-empty interior.

Suppose that \( f \in X \) and \( \varepsilon > 0 \). We found a piecewise linear function \( g \) with \( \| f - g \| \leq \varepsilon / 2 \).

Now we find \( h \) with \( \| g - h \| \leq \varepsilon / 2 \) and with \( h \) having slope \( > n \) everywhere. Consider \( \Psi \) as drawn, where \( M \) is a large \( \mathbb{N} \) integer and \( \Psi(\frac{j}{M}) = (-1)^{j+1} \), \( \Psi \) linear on each interval \( [\frac{j}{M}, \frac{j+1}{M}] \).

\( \Psi \) is continuous, and it has slope \( \geq 2M \) at every point in \( [0, 1] \).

Define \( h = g + \frac{\varepsilon}{2} \Psi \). Clearly \( \| g - h \| \leq \varepsilon / 2 \) is bounded. The slope of \( g \) is bounded by some \( C \) since \( g \) is piecewise linear. The slope of \( h \) is always at least \( \varepsilon / 2 \cdot 2M - C \).

Choosing \( n \) large enough, this can be made \( > n \). Thus, we have found \( h \notin A_n \), \( \| f - h \| \leq \varepsilon \). Since \( \varepsilon \) was arbitrary, \( A_n \) has empty interior. \( \Rightarrow \)

Uniform Boundedness Principle

Theorem

Suppose that \( X \) is a complete metric space. Suppose that \( F \) is a collection of continuous functions on \( X \). Suppose that

\[ \forall x \in X, \quad \sup_{f \in F} |f(x)| < \infty. \]

Then there is some ball \( B_{\delta}(x_0), \delta > 0 \), such that

\[ \sup_{f \in F} \sup_{x \in B_{\delta}(x_0)} |f(x)| < \infty. \]

(When \( X = [0, 1] \), this states that if \( |f(x)| \leq C_x \), for all \( f \in F \), for some \( C_x \), then the
functions are in fact uniformly bounded on some interval \((a, b)\).

**Proof**

For every \(n\), the set \( \bigcap_{x \in X} \{ x \in X : |f(x)| \leq n \} \) is closed.
Indeed, since \( f \) is continuous, \( f^{-1}([n, n]) = \{ x \in X : |f(x)| \leq n \} \) is closed, and arbitrary intersections of closed sets are closed.

By assumption, \( X = \bigcup_{n} A_n \). Thus by Baier, one of the \( A_n \) has non-empty interior.

**Points of Continuity**

or any countable, dense set

**Theorem**

Suppose \( f : [0, 1] \to \mathbb{R} \) is continuous at every rational. Then \( f \) is in fact continuous at uncountably many points of \([0, 1]\).

**Proof**

The key claim is that the set of points where a function is continuous is a \( G_\delta \) set, by which we mean a countable intersection of open sets.

To prove this, consider \( \alpha f(x) = \lim_{\delta \to 0} \sup_{0 < |\delta| < \delta} |f(x + \delta) - f(x)| \)

The limit exists since the 'variation' quantity is non-increasing in \( \delta \). We claim that \( \alpha f(x) = 0 \iff f \) is continuous at \( x \), an easy exercise. Furthermore, \( \{ x : \alpha f(x) < \varepsilon \} \) is open for all \( \varepsilon > 0 \). Indeed, suppose that \( \alpha f(x) < \varepsilon \).

If \( \varepsilon \) is small enough, \( \sup_{0 < |\delta| < \delta} |f(x + \delta) - f(x)| \leq \varepsilon' < \varepsilon \).

If \( |x' - x| \leq \frac{\delta}{2} \), then

\( \sup_{0 < |\delta| < \delta} |f(x) - f(y)| \leq \sup_{0 < |\delta| < \delta} |f(x + \delta) - f(y)| \leq \varepsilon' < \varepsilon \).
Hence, the set of points of continuity of \( f \) is \( \bigcap_{n \geq 1} \{ x : |(f(x))| < \frac{1}{n} \} \) is a \( G_\delta \) set.

**Claim**

\( \mathbb{Q} \cap [0, 1] \), the rationals in \([0, 1]\), is not a \( G_\delta \) set.

**Proof**

Suppose \( \mathbb{Q} \cap [0, 1] = \bigcap_{n \geq 1} \bigcup_{\mathbb{Q}} [q, 3] \cup \bigcup_{n \geq 1} U_n \) open.

Then \([0, 1] = \bigcup_{q \in \mathbb{Q}} [q, 3] \cup \bigcup_{n \geq 1} U_n \). This is a countable union of closed sets. Thus, by Baire, one of these has non-empty interior. Clearly \([q, 3]\) has empty interior. Alternatively, \( U_n \) contains an open ball. But this cannot happen, since \( \mathbb{Q} \) intersects every open ball. \( \square \)

**Banach-Tarski Paradox**

**Theorem**

Assume the Axiom of Choice. Then, we may divide the unit ball \( X \) in \( \mathbb{R}^3 \) into 17 pieces \( A_1, \ldots, A_{17} \) which may be rearranged to give two copies of \( X \). More precisely, there are isometries \( g_1, \ldots, g_{17} \in \text{Isom}^+(\mathbb{R}^3) \) (rotations and translations, orientation preserving) such that \( g_1 A_1 \cup \ldots \cup g_{17} A_{17} \) is two disjoint copies of \( X \).

**Remark**

There is no such example for \( \mathbb{R} \) or \( \mathbb{R}^2 \).

Why is this not really a paradox? There is no notion of
"weight" or volume valid for all subsets $A \subseteq X$. More mathematically,
there does not exist a finitely additive measure
$m : PX \rightarrow [0, \infty)$ such that
i) $\mu(gA) = \mu(A)$ for any isometry $g$.
ii) $\mu(A \cup B) = \mu(A) + \mu(B)$ if $A \cap B = \emptyset$
iii) $\mu(X) = \frac{4}{3} \pi$
**The Free Group on Two Generators**

This group consists of generators $a$, $b$, and all words in $a$, $b$, $a^{-1}$, $b^{-1}$ such that no word is equal to the identity unless it obviously is (e.g. $a b^{-1} b a^{-1}$ is obviously equal to the identity).

More formally, no reduced word such as $a^4 b^{-7} a^6$ is equal to the identity. For example, $a^3 b^6 b^{-2} a^{-3} a^{-3} b$ is not reduced, but reduces to $a^3 b^4 a^2 b$.

A picture of the free group, $F$, on two generators:

```
   e
  a a' b
 a^-1 b^-1
```

Write $F_a$ for the words beginning with $a$. Fairly obviously,

$$F = F_a \cup F_b \cup F_a^{-1} \cup F_b^{-1} \cup \{e\}$$

Note that $F_a \cup a F_a^{-1}$ is the whole of $F$, as is $F_b \cup b F_b^{-1}$.

Thus, the free group $F$ can be decomposed into 4 pieces (plus $e$) which can be rearranged to give two copies of $F$.

Key idea in Banach-Tarski: One can decompose the ball $B^3 \subset \mathbb{R}^3$ into tree-like structures. To do this, we will show that $\text{Isom}^+(\mathbb{R}^3)$ (and in fact the group $\text{SO}(3)$ of rotations in $\mathbb{R}^3$) contains a free subgroup.

Why is this not the case in $\mathbb{R}$ or $\mathbb{R}^2$?

$\text{Isom}^+(\mathbb{R}) \cong \mathbb{R}$ (translations). This group is abelian, and hence doesn't contain the free group $F$, since $ab a^{-1} b^{-1} = e$ always.

$\text{Isom}^+(\mathbb{R}^2)$ is generated by the group $\text{SO}(2)$ of rotations.
about 0 together with translations. Every such isometry has the
form \( x \mapsto Ax + b \), \( A \in \text{SO}(2) \), \( b \in \mathbb{R}^2 \).
This gives \( \text{Isom}^+(\mathbb{R}^2) \cong \text{SO}(2) \times \mathbb{R}^2 \) with
\((A, b) \star (A', b') = (A'A, A'b + b')\).
This group is not abelian. Indeed, if \( g_1 = (A, b) \), and
\( g_2 = (A', b') \), then \[ [g_1, g_2] = g_1 g_2 g_1^{-1} g_2^{-1} = (A'^{-1} A A', *) = (I, *) \], a translation, since
\( \text{SO}(2) \) is abelian. Hence \[ [g_1, g_2], [g_3, g_4] = e \]
for all \( g_1, \ldots, g_4 \), since any two translations commute.
Hence if \( \text{Isom}^+(\mathbb{R}^2) \) contained two elements \( a, b \) generating a
free group, we have \[ [a, b], [a^2, b^2] = e \]
not the trivial word.
This is a contradiction ( \( \text{Isom}^+(\mathbb{R}^2) \) is a solvable group).

**Theorem**

\( \text{SO}(3) \) contains two elements \( a, b \) generating a free group.

**Proof.**

\[
\begin{pmatrix}
\frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\
\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\
0 & 0 & 1
\end{pmatrix}
\quad
\begin{pmatrix}
0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \\
\frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \\
0 & 0 & 1
\end{pmatrix}
\]
righ rotations through angles \( \frac{\pi}{3} \) about the \( z \) and \( x \) axes respectively.

Suppose that there is a reduced word \( w \) which is the identity.
By considering \( a^{-m} w a^m \) for large \( m \), we assume that the
word \( w \) ends in \( a \).

Write \( w = g_n \ldots g_1 \), \( g_i \in \{ a, a^{-1}, b, b^{-1} \} \) and
\( g_i = a \). We will show that \( w\left(\frac{1}{3}\right) \neq w\left(\frac{2}{3}\right) \) and
derive a contradiction.
We have $g_n \ldots g_1 = \frac{1}{3^n \left( \frac{x_n y_n}{z_n} \right)}$ where $x_n, y_n, z_n \in \mathbb{Z}$.

By induction:

\[
(x_{n+1}, y_{n+1}, z_{n+1}) =
\begin{cases} 
(x_n - 4y_n, 2x_n + y_n, 3z_n), & g_{n+1} = c \\
(x_n + 4y_n, -2x_n + y_n, 3z_n), & g_{n+1} = a \\
(3x_n, y_n + 2z_n, -4y_n + z_n), & g_{n+1} = b \\
(3x_n, y_n - 2z_n, 4y_n + z_n), & g_{n+1} = b''
\end{cases}
\]

Write $(x_{n+1}, y_{n+1}, z_{n+1}) = \Phi_{g_{n+1}} (x_n, y_n, z_n)$.

Now consider the vectors $(x, y, z) \pmod{3}$. Let $\Phi_g$ be the corresponding map:

\[
\Phi_g (x, y, z) =
\begin{cases} 
(x - y, -x + y, 0), & g = a \\
(x + y, x + y, 0), & g = a'' \\
(0, y - z, -y + z), & g = b \\
(0, y + z, y + z), & g = b''
\end{cases}
\]

Consider the equivalence relation $(x, y, z) \sim (-x, -y, -z)$ (projective equivalence). The maps $\Phi_g$ act on these equivalence classes.

Consider the points $p_a = (-1, 1, 0)$, $p_{a''} = (1, 1, 0)$, $p_b = (0, 1, -1)$, $p_{b''} = (0, 1, 1)$ in this space.

Observe that $\Phi_g (p_a) = p_b$ unless $g' = g''$ for $g = a, a'', b, b''$ (proof by inspection).

$p_g$ is an attracting fixed point for $\Phi_g$. 
Note that $\Phi \left( \frac{1}{2} \right) = \rho_\alpha$.

Hence if $g_n \ldots g_1$ is a reduced word ending in $\alpha$, 
$\Phi g_n \Phi g_{n-1} \ldots \Phi g_1 \left( \frac{1}{2} \right)$ bounces around the diagram, but 
never ends up back at $\left( \frac{1}{2} \right)$

"Proof by ping-pong"
Theorem
There is a free subgroup on two (quite explicit) generators in $SO(3)$.

Remarks
i) There is actually a copy of $F_2$, the free subgroup on $n$ generators in $SO(3)$. In fact (not quite trivial) $F_2$ contains a copy of $F_n$.

ii) A random pair of elements in $SO(3)$ (whatever this means) will generate a free group. "Proof": Consider a word $w(a, b)$ such as $a^2b^7a^{-1}$. The set $w(a, b) = id$ is a proper subvariety of $SO(3)$, given by a polynomial. This is proper since there is a choice $a = a_0, b = b_0$ for which $w(a_0, b_0) = id$ as we saw last time. There are countably many words, and for each one, $P(w(a, b) = id) = 0$. Hence $P(w(a, b) = id$ for some $w$) = 0.

Hausdorff Paradox ("Baby" Banach-Tarski)

Theorem
There is a countable set $O \subset S^2 \subset \mathbb{R}^3$ such that $X = S^2 \setminus O$ has the following property: For disjoint $A_1, \ldots, A_\infty \subset X, a_1, \ldots, a_\infty \in SO(3)$ such that $g_1 A_1 \cup g_2 A_2 = g_3 A_3 \cup g_4 A_4 = X$.

Proof
Let $F$ be a free group on the generators $a, b$ in $SO(3)$. $F$ is countable. Let $O$ consist of the two (antipodal) fixed points of all rotations in $F$. Let $X = S^2 \setminus O$. Then $F$ acts on $X$ by rotations. $X$ splits up into orbits $\{g x : g \in F\}$ under this action.
Let $Y \subset X$ be a set containing precisely one point from each orbit. (Serious use of AC). Then $X$ is the disjoint union $U \cup U \cup \ldots \cup U_{y \in E}$. Recall from last time that we may split $F = F_a \cup F_b \cup F_b \cup F_b \cup \ldots$ such that $F = F_a \cup F_b \cup F_b \cup F_b \cup \ldots$

Now define $A_1 = U_{y \in E} \cup U_{y \in F_b \cup F_b \cup \ldots}$

$A_2 = U_{y \in F_a \cup F_b \cup F_b \cup \ldots}$

Take $g_1 = g_3 = id, g_2 = a, g_4 = b$

Proposition

There are disjoint sets $A_1, \ldots, A_8 \subset S^2$ and $g_1, \ldots, g_8 \in SO(3)$ such that $S^2 = g_1 A_1 \cup \ldots \cup g_4 A_4 = g_5 A_5 \cup \ldots \cup g_8 A_8$

Proof

Let $D$ be as in the Hausdorff paradox (we claim a countable union). We claim that $S^2 = g_1 A_1 \cup \ldots \cup g_4 A_4 = g_5 A_5 \cup \ldots \cup g_8 A_8$.

Consider $B_1 = S^2 \setminus (D \cup \rho D \cup \rho^2 D \cup \ldots) = B_1 \cup \rho B_1 \cup \rho^2 B_1 \cup \ldots$

Let $A_1, \ldots, A_8, g_1, \ldots, g_8$ be as in the Hausdorff paradox.

Define $C_{ij} = A_i \cap g_j^{-1} B_j, i = 1, \ldots, 4, j = 1, 2$.

These 8 sets are disjoint.
Coding and Cryptography

\[ S^2 = \mathcal{C}_1 \cup \mathcal{C}_2 \cup g_2 \mathcal{C}_2 \cup g_1 \mathcal{C}_1 \cup g_2 \mathcal{C}_2 \]

This is \( 8 \) sets, \( 8 \) rotations of the type claimed.

**Proposition**

There are disjoint sets \( A_1, \ldots, A_8 \subset S^2 \) and \( g_1, \ldots, g_8 \in SO(3) \) such that 

\[ S^2 = g_1 A_1 \cup \ldots \cup g_4 A_4 = g_5 A_5 \cup \ldots \cup g_8 A_8 \]

**Proof**

As above with \( S^2 \) replaced by \( B^3 \setminus \{0\} \). Decompose into spherical shells.

**Proposition**

There are disjoint sets \( A_1, \ldots, A_{16} \subset B^3 \) and \( g_1, \ldots, g_{16} \in Iso^+(\mathbb{R}^3) \) such that 

\[ B^3 = g_1 A_1 \cup \ldots \cup g_{16} A_{16} \]

**Proof**

Let \( \Theta \) be an irrational rotation about \( (0, 0, \frac{1}{2}) \).

Define \( B_1 = B^3 \setminus \{0, \Theta(0), \Theta^2(0), \ldots\} \). 

\( B_2 = \{ \Theta(0), \Theta^2(0), \ldots\} \) are distinct points in \( B^3 \).

Note that \( B_1 \cup B_2 = B^3 \setminus \{0\} \) and \( B_1 \cup \Theta^{-1}(B_2) = B^3 \).

Now repeat what was done before.

**Theorem**

We can decompose \( B^3 \) into \( 17 \) disjoint sets which may be rearranged to form two copies of \( B^3 \).
Proof

Let $X = A_1 \cup \cdots \cup A_{16}$. Consider the maps $f : X \to B^3 \cup B^3$, $\Psi : B^3 \to B^3 \cup \emptyset$

Set $S_1 = f^{-1}(\Psi(S_0))$

$S_2 = f^{-1}(\Psi(S_1))$ etc.

Define $S = \bigcup_{i=0}^\infty S_i$

Consider $A_i' = A_i \cap S$ for $i = 1, \ldots, 16$, and $A_{17} = S$. This is a decomposition of $B^3$ into 17 pieces and $f(x(S)) = (B^3 \cup B^3) \setminus \Psi(S)$.

Thus $A_1', \ldots, A_{16}'$ rearrange to form $(B^3 \cup B^3) \setminus \Psi(S)$.

$S$ is a rigid translate of $\Psi(S)$

(See Cantor-Schröder-Bernstein, "back and forth")
The Brunn-Minkowski inequality and Isoperimetric inequality

We will understand that we understand the volume of open
subsets of $\mathbb{R}^n$ (using only $n = 2$). We will write $|V| = \text{vol}(V)$

**Theorem (Brunn-Minkowski)**

Let $A, B \subseteq \mathbb{R}^n$, open and bounded. Define $A + B$ to be
\[
\{a + b \mid a \in A, b \in B\} \quad \text{(easily seen to be open)}.
\]
Then
\[
|A + B|^n \geq |A|^n + |B|^n.
\]

**Remark**

We have equality if $A, B$ are convex.

**Isoperimetric Inequality**

Suppose $A \subseteq \mathbb{R}^n$. We define the Minkowski Surface Area $|\partial A|$ as follows: Let $S \subseteq \mathbb{R}^n$ be the unit ball, and define
\[
|\partial A| = \lim_{\varepsilon \to 0^+} \frac{|A + \varepsilon S| - |A|}{\varepsilon}.
\]
When $A$ is "nice" this corresponds well with the intuitive notion of surface area.

For notational simplicity (more or less the only reason), set $n = 2$.

**Theorem (Isoperimetric Inequality)**

Let $A \subseteq \mathbb{R}^2$. Then
\[
|\partial A| \geq \frac{|\partial S|}{|S|}.
\]
(Suitably normalised, nothing has smaller surface area than $S$).

**Proof**

We have
\[
A + \varepsilon S - |A| \geq \frac{(|A| + \varepsilon |S|)^2 - |A|}{\varepsilon}
\]
(Brunn-Minkowski with $B = \varepsilon S$. Note $|B| = \varepsilon^2 |S|$)

Hence, expanding out,
\[
|\partial A| = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} (|A + \varepsilon S| - |A|)
\]
\[
\geq \lim_{\varepsilon \to 0} \left(2|A|^{1/2} |S|^{1/2} + \varepsilon |S| \right) = 2|A|^{1/2} |S|^{1/2}.
\]
On the other hand, \( |2S| = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left( \left| \frac{15 + \varepsilon S - 15}{\varepsilon} \right| \right) \).

(But \( S + \varepsilon S = (1+\varepsilon)S \) = \( \lim_{\varepsilon \to 0} \frac{(1+\varepsilon)^2 - 1}{\varepsilon} |S| = 2|S| \).

Thus \( \frac{1a_{11}}{1a_{11}} \geq 2|S|^{1/2} = \frac{1051}{151^{1/2}} \).

**Proof of Brunn-Minkowski**

Idea: Verify the case when \( A, B \) are finite unions of boxes, by induction on the number of boxes.

**Base Case**

\( A \) and \( B \) are both boxes. Suppose that \( A \) has dimensions \( a_1, a_2 \), and \( B \) has \( b_1, b_2 \). Then \( |A + B| = (a_1 + b_1)(a_2 + b_2) \).

\[ |A| = a_1a_2, \quad |B| = b_1b_2. \]

We need to check that \( |A + B| = (a_1 + b_1)(a_2 + b_2) \geq \left( \frac{a_1a_2 + b_1b_2}{2} \right)^2 \).

Rearranging, this is equivalent to \( a_1b_2 + b_1a_2 \geq 2 \sqrt{a_1a_2b_1b_2} \).

i.e. \( (a_1b_2 - b_1a_2)^2 \geq 0 \). (Remark: use AM-GM and other inequalities for the \( n \) dimensional case).

**Inductive Step**

Suppose that \( A \) has \( \geq 2 \) boxes (horizontal or vertical) lie in \( \mathbb{R}^2 \) that divides \( A \) properly, so that if \( A \) has \( n \) boxes, then there are fewer than \( n \) boxes both below and above the line, i.e. then \( A^+, A^- \) have strictly fewer boxes than \( A \), where

\[ A^+ = \bigcup \text{union of boxes above the line} \]
\[ A^- = \bigcup \text{union of boxes below the line} \]
Now move $B$ up or down if necessary in such a way that
\[
\frac{IA_+}{IA} = \frac{IB^+}{IB}
\]
where $B^+$ is the intersection of $B$ with the points above $L$. This is possible by continuity: if $B$ is moved to $+\infty$ this ratio $\to 1$ and if $B$ is moved to $-\infty$, the ratio $\to 0$, so apply the Intermediate Value Theorem.

Now $IA + IB \geq IA_+ + B_+ + IA_+ + B_-$ (since $A_+ + B_+,$ $A_+ + B_-$ lie on opposite sides of $L+L$, so are disjoint.

Both $A_+ \cup B_+$ and $A_- \cup B_-$ have fewer boxes than $A \cup B$ since $A_+$, $A_-$ have fewer boxes than $A$.

Applying the induction hypothesis, this is at least
\[
(IA_+^2 + IB^2) + (IA_+^2 + IB_-^2)
\]
\[
= IA_+ \left(1 + \frac{IB^+}{IA_+}\right)^2 + IB_+ \left(1 + \frac{IB_-}{IA_-}\right)^2 \tag{*}
\]

However \(IA_+ \neq IB\) and so \(IA_- \neq IB\) so $IA_+ = 1 - \frac{IB^+}{IB} = 1 - \frac{IB^+}{IB} = \frac{IB}{IA}$

and so \(IA_- = IB_- = \frac{IB}{IA}\)

Hence (*) simplifies to
\[
(IA_+ + IA_-) \left(1 + \frac{IB}{IA}\right)^2 = (\sqrt{IA} + \sqrt{IB})^2
\]
QED.
Topics in Analysis
Basics of Sets and the Kakeya Problem

Kakeya Problem

What is the area of the smallest set \( E \subset \mathbb{R}^2 \) in which one can rotate a (thin) unit rod through 180°?

Answer

For every \( E > 0 \) there is a set \( E \) with this property and \( 1E1 < E \).

We will not quite show this, but will construct a closely related object.

Theorem (Besicovitch)

There is a compact set \( E \subset \mathbb{R}^2 \) which contains a unit line segment in every direction but has measure 0.

Remark

It is 'easy' to specify a set \( E \) with this property, but surprisingly hard to show that it has measure 0.

\[
C = \{ \text{base 4 Cantor Set} \} = \left\{ \sum_{i=1}^{\infty} a_i 4^{-i} : a_i \in \{0, 1\} \right\}
\]

\[
2C = \left\{ \sum_{i=1}^{\infty} b_i 4^{-i} : b_i \in \{0, 2\} \right\}
\]

Join each point on the bottom to every point on the top with a straight line. Since everything in \([0, 1]\) can be written as a difference of something in \(C\) and something in \(2C\), the resulting set has a unit line segment of every angle \( \theta \) within \( \frac{\pi}{2} \) of vertical. The union of 10 rotations of this set has a line segment in every direction. It is true, but tricky to show, that this set has measure 0.
Geometric Lemma

Let $T$ be a triangle with base on the real line and with height 1. Bisect the base, giving two triangles $T_1$ and $T_2$. Move $T_2$ an amount $\delta \delta$ to the left as shown:

This gives a new triangle with area $(1-\delta)^2 |T_1|$. This is similar to $T$ but the base is shrunk by a factor $1-\delta$. We also have the red "bowtie" shown, which has area $2\delta^2 |T_1|$ (To prove this, dilate so that $T$ is an isosceles right triangle and then use Euclidean geometry).

Lemma

Start with a triangle $T$, contained in an open set $V$. Let $\eta > 0$. Then, for some $k$, we can divide the base of $T$ into $2^k$ parts, which we can then slide around to give a set $E$, which is a union of $2^k$ triangles, with total area less than $\eta$.

Furthermore, we can assume that $E = V$.

Proof

Let $\delta = \frac{\eta}{10}$. Let $k$ be a quantity to be specified later. Suppose WLOG that $|T_1| = 1$. Divide the base of $T$ into $2^k$ equal parts, $T_1, \ldots, T_{2^k}$. Apply the procedure from the Geometric Lemma to the pairs $T_1, T_2$, then $T_3, T_4$, and so on. The total area of the "bowties" is $\delta^2$.
Topics in Analysis

We have new triangles $T_1$, $T_2$, ..., $T_{2^k-1}$ of total area $\frac{1}{2} (1 - \delta)^2$. Now we apply the same procedure to $T_1$ and $T_2$, $T_3$ and $T_4$, and so on, moving the "bowties" with the triangles. Repeat this until it has been done $k$ times in total.

At the end of the process, we have a single triangle of area $\frac{1}{2} (1 - \delta)^{2k}$, and a union of "bowties", with total area at most $S^2 \left( 1 + (1 - \delta)^2 + (1 - \delta)^4 + ... + (1 - \delta)^{2k} \right) \leq \frac{S^2}{(1 - \delta)^2} \leq 4S$

if $\delta$ is small. Choosing $k$ large enough, we can make the total area $\leq \eta$. Furthermore, the resulting set has a unit line segment in all the directions that $T$ did.

Finally, we remark on how this can be done while staying in $V$. Since $T$ is compact and $V^c$ closed, some neighborhood $N_\varepsilon (T)$ also lies in $V$. At the beginning of the construction, divide $T$ into finitely many triangles of base $< \varepsilon$. Then perform the construction as described, noting that nothing was moved by more than $\varepsilon$.

Conclusion of Proof

Begin with a right isosceles triangle $T$. This contains a unit line segment for each angle $\theta \in [0, \frac{\pi}{4}]$. Put $T$ inside some open $V_0$. Now perform the construction, to get a union of triangles $T_i \subseteq V_0$, and area $< \frac{1}{15}$. Let $V_i$ be an open subset containing $T_i$ and with $V_i \subseteq V_0$. Now repeat the construction
on $T_i$, getting a union of triangles $T_2 \subset V_i$, $|T_2| < \frac{1}{100}$.
Continue in this fashion, ensuring that $|V_i| \leq 2|T_i|$, no
$|V_i| \to 0$ as $i \to \infty$.
Finally, let $F = \cap V_i$. We claim that $F$ contains a unit line segment in every direction $\Theta \in [0, \frac{\pi}{4}]$. By construction, each $T_i$, and hence $V_i$, contains a line segment $x_i + e^{2\pi i \Theta} [0, 1]$.
By sequential compactness, we may pass to a subsequence with $x_i \to x$. Since $F$ is closed, it contains $x + e^{2\pi i \Theta} [0, 1]$.
Since $F \subset V_i$, $|V_i| \to 0$, $F$ has measure 0. Finally, take 8 rotated copies of $F$.

By taking products of planar Besicovitch sets with $d - 2$ dimensional planes, we can obtain similar examples in $\mathbb{R}^d$.

Unsolved Problem

Does a Besicovitch set in $\mathbb{R}^d$ have Minkowski dimension $d$?

$$\sup \left\{ \frac{\varepsilon}{d} \left| N_{\varepsilon}(E) \right| \right\} \leq \sup \left\{ \frac{S}{\varepsilon^d} \left| N_{\varepsilon}(S) \right| \right\} = 0$$

This is solved when $d = 2$. When $d = 3$, the best bound is $\frac{3}{2} + 10^{-10}$ (Katz, Laba, Tao)