Ramsey Theory

"Can we find some order inside sufficient disorder?"

Chapter
1. Monochromatic Systems
2. Partition Regular Equations
3. Infinite Ramsey Theory

Books
1. Bollobás, "Combinatorics", CUP 1986, for Chapter 3
2. Graham, Rothschild, Spencer, "Ramsey Theory", Wiley 1971

For chapters 1 and 2.

Chapter 1: Monochromatic Systems

Write \( N = \{1, 2, 3, \ldots \} \).

For a set \( X \), we write \( X^{(r)} \) for \( \{ A \subseteq X : |A| = r \} \).

Given a 2-colouring of \( \binom{N}{2} \) (i.e., a function \( C: \binom{N}{2} \rightarrow \{1, 2\} \)),
can we always find an infinite monochromatic set \( M \)?
(i.e., infinite \( M \subseteq N \) such that \( C \) is constant on \( M^{(2)} \)).

We write \( i \rightarrow j \) for \( \{i, j\} \), the edge from \( i \) to \( j \), where \( i < j \).

Examples
1. Colour is \textit{red} if \( i + j \) is \textit{even}.
   Then \( M = \{2, 4, 6, \ldots \} \) is monochromatic (red).
2. Colour is \textit{blue} if \( \max \{ n : 2^n \mid (i+i) \} \) is \textit{odd}.
   Then \( M = \{4^0, 4^1, 4^2, \ldots \} \) is monochromatic.
3. Colour is red if it has an even number of distinct prime factors. No examples of Man is known. However...

**Theorem 1** (Ramsey's Theorem)

Whenever $\mathbb{N}^{(2)}$ is 2-coloured, there exists an infinite monochromatic set.

**Proof**

Choose $a_1 \in \mathbb{N}$ (any will do). There are infinitely many edges from $a_1$, so infinitely many are the same colour (pigeonhole principle), so there is a set $B_1$ such that all edges from $a_1$ to $B_1$ are colour $C_1$, and $B_1$ is infinite.

Choose any $a_2 \in B_1$.

There are infinitely many edges from $a_2$ to $B_1 \setminus \{a_2\}$, so again, there exists infinite $B_2 \subset B_1 \setminus \{a_2\}$ and $C_2$ such that all edges from $a_2$ to $B_2$ have colour $C_2$.

Continue inductively.

We obtain $a_1, a_2, a_3, \ldots$ and colours $C_1, C_2, C_3, \ldots$ such that $\forall i$, all edges $a_i; a_j$ ($i < j$) have colour $C_i$.

But then, infinitely many of the $C_i$ are the same, say $C_{i_1} = C_{i_2} = C_{i_3} = \ldots$. Then $\{a_{i_1}, a_{i_2}, a_{i_3}, \ldots\}$ is monochromatic.

$\square$
Remarks

1. Called a '2-pass' proof.

2. The same proof shows that whenever $\mathbb{N}^{(2)}$ is $k$-coloured (i.e. we have $c : \mathbb{N}^{(2)} \rightarrow [k]$) so there exists an infinite monochromatic $\mathbb{N}$. Alternatively, view the colours as '1' and '2 or 3 or ... or $k$' and apply theorem 1, getting an infinite set of colour 1 (done) or with colours 2, 3, ..., $k$, so we are done by induction.

3. Having an infinite monochromatic set is stronger than asking for arbitrarily large finite monochromatic sets.

Example

Any sequence $x_1, x_2, ...$ in $\mathbb{R}$ (or any totally ordered set) has a monotone subsequence. Indeed, 2-colour $\mathbb{N}^{(2)}$ by giving $i$'s colour down if $x_i < x_i$ and apply theorem 1.

What about $\mathbb{N}^{(r)}$, $r = 3, 4, ...$? If we 2-colour $\mathbb{N}^{(r)}$, do we get an infinite monochromatic set?

e.g. $r = 3$: colour $\mathbb{N}^{(3)}$ by giving $i \in [3] (i < j < k)$ colour red if $i \in [j \cup k]$ blue if $i \not\in [j \cup k]$.

We could take $M = \{2, 4, 8, 16, ... \}$

Theorem 2 (Ramsey for $r$-sets)

Whenever $\mathbb{N}^{(r)}$ is 2-coloured, there exists an infinite monochromatic set.
Proofs

By induction on $r$: $r = 1$ is the Pigeonhole principle.

$r = 2$ is Theorem 1.

Given a 2-colouring $c$ of $N^{(2)}$, choose $a_1 \in N$.

Choose $a_1 \in N$. We induce a 2-colouring of $(N \setminus \{a_1\})^{(r-1)}$ by $c'(F) = c(F \cup \{a_1\})$. By induction, we have an infinite monochromatic $B_1$ for this colouring. So all $r$-sets $F \cup \{a_1, \ldots, F \cup B_1$ have the same colouring, $C_1$ say. Choose $a_2 \in B_1$. By the same argument, there exists an infinite monochromatic $B_2 \subset B_1 \setminus \{a_1\}$ such that all $r$-sets $F \cup \{a_1, a_2\}$, $F \cup B_2$ have the same colour, $C_2$ say. Continue inductively. We obtain points $a_1, a_2, \ldots$

such that each $r$-set $\{a_i, \ldots, a_{i+1}\}$ $(i < \ldots < r)$ has colour $C_i$.

But we have $C_i = C_{i+1} = \ldots$ for some subsequence (by the Pigeonhole principle), whence $M = \{a_i, a_{i+1}, \ldots\}$.

Example

We saw that given given points $(1, x_1)$, $(2, x_2)$, $(3, x_3)$ we can find a subsequence such that the induced (piecewise-linear) function is monotone.

In fact, we can insist that the induced function is convex or concave. Indeed, 2-colour $N^{(3)}$ by giving $i \leq k$ colour concave if $x_i < x_k$ and apply Theorem 2.
Surprisingly, Infinite Ramsey (Theorem 2) implies the finite version:

**Theorem 3**

\[ \forall m, r, \exists n \text{ such that } [m]^r_n \text{ is 2-coloured, there exists a} \]

monochromatic set of size \( n \).

**Proof**

Suppose not, so that \( \forall n > r, \) there exists a 2-colouring \( c_n \) of \( [m]^r_n \) without a monochromatic set of size \( m \). We will construct a 2-colouring of \( \mathbb{N}^r \) without a monochromatic set of size \( m \), contradicting Theorem 2 (very strongly).

[If the \( c_n \) are nested, i.e. \( c_n \mid [n-1]^r_n = c_{n-1} \) we can take the union, but they may not be...]

There are only finitely many ways to 2-colour \( [r]^r \) (2 ways). So infinitely many of the \( c_n \) agree on \( [r]^r \), say \( c_n \mid [r]^r = c_r, \forall n \in B_1 \).

There are only finitely many ways to 2-colour \( [r+1]^r \), so infinitely many of the \( c_n, n \in B_2 \), agree on \( [r+1]^r \); say \( c_n \mid [r+1]^r = c_{r+1}, \forall n \in B_2 \).

Continue inductively. We obtain \( d_r, d_{r+1}, \ldots \) where

\[ d_r : [r]^r \rightarrow \{1, 2\} \text{ such that:} \]

i) The \( d_i \) are nested.

ii) No \( d_2 \) has a monochromatic \( m \)-set (as \( d_n = c_n \mid [m]^r_n \), \( m \neq n \)).

Now define \( c : \mathbb{N}^r \rightarrow \{1, 2\} \) by setting \( c(F) = d_n(F) \) for any \( n \leq \max F \). Then \( c \) has no monochromatic set of size \( m \). \( \square \)
1. The proof gives no bounds on what $n = n(m, r)$ we could take. There are direct proofs, that do give upper bounds.

2. Called a ‘compactness argument’. Essentially, we are proving that the space $\{0, 1\}^N$ (all 0–1 sequences) with the product topology (i.e. metric $d(f, g) = \frac{1}{\min \{n : f(n) \neq g(n)\}}$) is (sequentially) compact.

What if we coloured $\mathbb{N}^{(2)}$ with infinitely many colours — i.e. we have $c : \mathbb{N}^{(2)} \to X$ for some set $X$.

Obviously we cannot find an infinite $M$ on which $c$ is constant, e.g. let $c$ be injective (give every edge a different colour).

Can we always find infinite $M$ such that $c$ is either constant on $M^{(2)}$ or injective?

No, colour every edge is $(i < j)$ colour $c_i$.

**Theorem 4 (Canonical Ramsey Theorem)**

Let $c : \mathbb{N}^{(2)} \to X$ for some set $X$. Then $\exists$ infinite $M \subseteq \mathbb{N}$ such that one of the following holds:

i) $c$ constant on $M^{(2)}$

ii) $c$ injective on $M^{(2)}$

iii) $c(i, j) = c(k, l) \iff i = k \ (i, j, k, l \in M, \ i < j, k < l)$

iv) $c(i, j) = c(k, l) \iff j = l \ (i, j, k, l \in M, \ i < j, k < l)$

**Note**

This generalizes Theorem 1. For finite $X$ then ii), iii), iv) cannot arise.
Proof

2-colour $M^{(4)}$ by giving $i$ and $k$ the same colour if $c(ij) = c(kl)$ and different if not. By Ramsey for 4-sets (Theorem 2), there is an infinite monochromatic $M_1$ for this colouring.

If $M_1$ is colour same:

For any $i$ and $k$ in $M^{(2)}$, choose $mn \in M^{(2)}$ with $m > n$. Then $c(ij) = c(mn)$ and $c(kl) = c(mn)$. Therefore $c(ij) = c(kl)$ so $c$ is constant on $M^{(2)}$, case i).

So we may assume that $M_1$ has colour different.

Now 2-colour $M_2^{(4)}$ by giving $i$ and $k$ the same colour if $c(ij) = c(kl)$ and different if not.

By Theorem 2, we have infinite monochromatic $M_2 \subseteq M_1$ (monochromatic for this new colouring).

If $M_2$ has colour same:

Choose $i < j < k < l < m < n$ in $M_2$. Then $c(ij) = c(in)$ and $c(lm) = c(in)$, whence $c(ij) = c(lm)$ since $M_2 \subseteq M_1$. Thus $M_2$ is colour different.

2-colour $M_3^{(4)}$ by giving $i$ and $k$ the same colour if $c(ij) = c(kl)$ and different otherwise.

We have infinite monochromatic $M_3 \subseteq M_2$ for this colouring.

If $M_3$ has colour same, choose $i < j < k < l < m < n$ in $M_3$. Then $c(ij) = c(in)$ and $c(il) = c(km)$, so $c(in) = c(km)$, so $M_3$ is colour different.
2-colour $M_3^{(3)}$ by giving ijk colour same if $c(ij) = c(ik)$ and diff if not.

We have infinite monochromatic $M_4 \subset M_3$ for this colouring.

If $M_4$ has colour same:

Choose $i < j < k < l$ in $M_4$. Then $c(ij) = c(ik) = c(ik)$.

So $M_4$ is colour diff.

Now 2-colour $M_4^{(3)}$ by giving ijk colour left same if $c(ij) = c(ik)$, left-diff if not.

We get infinite monochromatic $M_5 \subset M_4$ for this.

Then 2-colour $M_5^{(3)}$ by giving ijk colour right same if $c(ik) = c(ik)$, right-diff if not. Infinite mono $M_6$ for this.

If $M_6$ left-same right-diff: Case ii)

If $M_6$ left-same right-diff: Case iii)

If $M_6$ left-diff right-same: Case iv)

If $M_6$ left-same right same:

Choose $i < j < k$ in $M_6$. Then $c(ij) = c(ik) = c(ik)$.

Remarks

1. We could use just one colouring according to the pattern of colourings of the 2-sets inside the given four-set.

2. For any r, we can show similarly:

For any colouring $c$ of $N^{(r)}$, 3 infinite monochromatic $M \subset N$ and I = [I] such that $i, j, \ldots, k$ in $M$, $c(i, \ldots, j, r) \Rightarrow c(i, \ldots, j, r)$ $\Rightarrow$ $in = in + bn \in I$.
These $2^n$ colorings are called the canonical colorings of $N^{(n)}$.

E.g. $r = 2$, $I = \{1\}$ is case iii)
$I = \{2\}$ is case iv)
$I = \{1, 2\}$ is case ii)
$I = \emptyset$ is case i)
Remarks

1. The proof gives no bounds on what $n = n(M, r)$ we could take. There are direct proofs, that do give upper bounds.

2. Called a 'compactness argument'. Essentially, we are proving that the space $\{0, 1\}^\mathbb{N}$ (all 0-1-sequences) with the product topology (i.e. metric $d(f, g) = \frac{1}{\min\{n : f(n) \neq g(n)\}}$) with the product is (sequentially) compact.

What if we coloured $\mathbb{N}^{(2)}$ with infinitely many colours? i.e. we have $c : \mathbb{N}^{(2)} \to X$ for some set $X$.

Obviously, we cannot find an infinite $M$ on which $c$ is constant. e.g. let $c$ be injective (give every edge a different colour).

Can we always find infinite $M$ such that $c$ is either constant on $M^{(2)}$ or injective?

No, colour every edge $i, j$ (i < j) colour $c_i$.

Theorem 4 (Canonical Ramsey Theorem)

Let $c : \mathbb{N}^{(2)} \to X$ for some set $X$. Then $\exists$ infinite $M \subset \mathbb{N}$ such that one of the following holds:

- i) $c$ constant on $M^{(2)}$
- ii) $c$ injective on $M^{(2)}$
- iii) $c(i, j) = c(k, \ell) \Leftrightarrow i = k$ \& $j < \ell \leq M$, $\ell < j, k < \ell$
- iv) $c(i, j) = c(k, \ell) \Leftrightarrow j = \ell$ \& $i, j, k, \ell \leq M$, $i < j, k < \ell$

Note

This generalizes Theorem 1. For finite then ii), iii), iv) cannot arise.
Proof

2-colour $N^{(4)}$ by giving $i$ and $k$ the same colour otherwise. By Ramsey for 4-sets (Theorem 2), there exists an infinite monochromatic $M_i$ for this colouring.

If $M_i$ is colour same:

For any $i < s < k < l$ in $M_i$, choose $mn \in M_i$ with $M > 3, l$. Then $c(i) = c(mn)$ and $c(kl) = c(mn)$. Therefore $c(i) = c(kl)$ so $c$ is constant on $M_i$, case i).

So we may assume that $M_i$ has colour diff.

Now 2-colour $M_i^{(4)}$ by giving $i$ and $k$ the same colour otherwise. By Theorem 2, we have infinite monochromatic $M_2 \subseteq M_i$ (monochromatic for this new colouring).

If $M_2$ has colour same:

Choose $i < j < k < l < m < n$ in $M_2$. Then $c(ij) = c(in)$ and $c(km) = c(in)$, whence $c(ik) = c(km)$ since $M_2 \subseteq M_i$. Thus $M_2$ is colour diff.

2-colour $M_2^{(4)}$ by giving $i$ and $k$ the same colour otherwise. We have infinite monochromatic $M_3 \subseteq M_2$ for this colouring.

If $M_3$ has colour same, choose $i < j < k < l < m < n$ in $M_3$. Then $c(il) = c(in)$ and $c(km) = c(km)$, so $c(in) = c(km)$ since $M_3 \subseteq M_2$.
We aim to prove that whenever $\mathbb{N}$ is 2-coloured, there exist a monochromatic arithmetic progression of length $m$, for any $m$. E.g. $\{a, a+d, \ldots, a+(m-1)d\}$, length $m$ ($n$ member of set).

By Compactness, this is the same as:

$\forall m \exists n$ such that when $[n]$ is 2-coloured $\Rightarrow \exists$ monochromatic arithmetic progression of length $m$.

Indeed, if not, then $\forall n \exists$ a 2-colouring $c_n$ of $[n]$ without a monochromatic arithmetic progression of length $m$. We have infinitely many $c_n$ agreeing on $[1]$, and of these, infinitely many agree on $[2]$. Continuing, we obtain a 2-colouring of $\mathbb{N}$ without a monochromatic arithmetic progression of length $m$.

One key idea in the proof is to show that $\forall m, k, \exists n$ such that $[n]$ $k$-coloured contains a monochromatic arithmetic progression of length $m$.

(A harder result could be easier to prove, if the proof is by induction)

Write $W(m,k)$ for the least such $n$ (if it exists). This is referred to as a "van der Waerden number".

Let $A_1, \ldots, A_r$ be arithmetic progressions of length $m-1$, say $A_i = \{a_i, a_i+d, \ldots, a_i+(m-2)d\}$

We say that $A_1, \ldots, A_r$ are focussed at $f$ if $a_i + (m-1)d = f$

E.g. $[1,43]$ and $[5,63]$ are focussed at $7$.

If each $A_i$ is monochromatic (for a given colouring) with no two
At the same colour, we say that $A_1, \ldots, A_n$ are colour-focused.

(prof. if we have any $r$-colouring and $A_1, \ldots, A_n$ are colour-focused then we get a monochromatic arithmetic progression of length $m$, by asking "What colour is the focus?"

**Proposition 5** (contained within Theorem 6)

For all $k$, if in each that whenever $[n]$ is $k$-coloured, there exists a monochromatic arithmetic progression of length 3.

**Proof**

1. *k ≤ r ≤ k*

We claim there exists such that whenever $[n]$ is $k$-coloured, $A$ either

- A monochromatic arithmetic progression of length 3
- OR
- r colour-focused arithmetic progressions of length 2.

(Then we are done by setting $r = k$ and looking at the focus)

**Proof by induction on $r$**: ($r = 1$ is easy, $r = k+1$)

We will show that if $n$ is suitable for $r-1$, then $(k^{2n+1})2n$ is suitable for $r$.

Indeed, given a $k$-colouring of $[k^{2n+1}]$ with no monochromatic arithmetic progression of length 3, we break $[k^{2n+1}]$ into intervals $B_1, \ldots, B_{k^{2n+1}}$ of length $2n$: $B_i = [2n(i-1)+1, 2ni]$.

Now, there are $k^n$ ways to $k$-colour a block. There are $k^{2n+1}$ blocks. Hence WLOG, $B_5$ and $B_{5+k}$ are coloured identically.

By choice of $n$, inside $B_5$, we have $r-1$ colour-focused arithmetic progressions of length 2, together with their focus.
say \{a_i, a_i + d, \ldots, a_i + d(n-1)\} concentrated at 1.
But now \{a_i, a_i + d, \ldots, a_i + 2nt\}, 1 \leq i \leq r-1 are colour concentrated at 1 + 4nt and also \{1, 1 + 2nt\} is monochromatic of a different colour. This gives 1 colour concentrated arithmetic progressions of length 2.

\[ \square \]

**Remark**

1. The idea of looking at "patterns of whole blocks" is called a product argument.

2. The proof gives bounds of the form $W(3, k) \leq k^{k^{k^k}}$ terms.
   This is called a "Towers-type" bound.
Theorem 6 (van der Waerden's Theorem)

\[ \forall m, k \in \mathbb{N} \text{ such that whenever } [n] \text{ is } k \text{-coloured, there exists a monochromatic arithmetic progression of length } m. \]

Proof

Induction on \( m \) (for all \( k \)). \( m = 1 \) is trivial (or \( m = 2 \) is true by the Pigeonhole principle, or \( m = 3 \) is Proposition 5).

Given \( m \), we may assume that \( W(m-1, k) \) exists for all \( k \). We claim that \( \forall 1 \leq r \leq k \exists n \text{ such that whenever } [n] \text{ is } k \text{-coloured we have either:} \)

i) A monochromatic arithmetic progression of length \( m \), or

ii) \( r \)-colour-focused arithmetic progression of length \( m-1 \).

(Then we are done by setting \( r = k \) and looking at the focus.)

Proof of claim: (induction on \( r \))

\( r = 1 \) is the (taking \( n = W(m-1, k) \))

Given \( n \) suitable for \( r-1 \), we will show that \( W(m-1, k^{2^n})2n \)

is suitable for \( r \).

Indeed, given a \( k \)-colouring of \( [W(m-1, k^{2^n})2n] \) without a monochromatic arithmetic progression of length \( m \), we break \( [W(m-1, k^{2^n})2n] \) up into blocks of length \( 2n \):

\[ B_1, B_2, \ldots, B_{W(m-1, k^{2^n})} \text{, where } B_i = [2n(i-1)+1, 2ni] \]
Each block may be coloured in $k^{2n}$ ways, so by definition of $W(m-1, k^{2n})$, we have $m-1$ equally spaced blocks with identical colourings; say blocks $B_s, B_{s+t}, ..., B_{s+(m-2)t}$.

Inside $B_s$, we have $r-1$ colour-focused arithmetic progressions of length $m-1$ (by choice of $n$). Together with their focus (as the blocks are length $2n$), progressions $A_1, ..., A_{r-1}$ focused at $f$, where $A_i = \{a_i, a_i + d, ..., a_i + (m-2)d\}$.

But then $\{a_i, a_i + (d+2nt), ..., a_i + (n-2)(d+2nt)\}$, for $1 \leq i \leq r-1$ are colour-focused at $f + (m-1)2nt$.

Also, if $f, f+2nt, ..., f+(n-2)2nt$ is monochromatic, of a different colour, so we have $r$ arithmetic progressions of length $m-1$, colour-focused at $f + (m-1)2nt$.

The Ackermann or Grzegorczyk hierarchy is the sequence of functions $f_1, f_2, ..., \ (\text{each } \mathbb{N} \rightarrow \mathbb{N})$ defined by:

$f_1(x) = 2x$
$f_{n+1}(x) = f_n^{(x)}(1) = f_n(f_n(\ldots(f_n(1))\ldots))$

e.g. $f_2(x) = 2^x$
$f_3(x) = 2^{2^\ldots^x}$ (height $x$)
$f_4(1) = 2$, $f_4(2) = 2^2 = 4$, $f_4(3) = 65536$
$f_4(4) = 2^{2^2} = 65536$, $f_4(5) = 2^{2^{2^2}2} = 65536$

We say that $f: \mathbb{N} \rightarrow \mathbb{N}$ is of type $\alpha$ if $f$ is a function with $f_n(\alpha x) \leq f(\alpha x) \leq f_n(\beta x)$ for all $x$. Our bound on $W(3, k)$ is of type 3.
For each $m$, our bound on $W(m, k)$ is of type $m^k$. Then, our bound on $W(m, 2) = W(m)$ grows faster than every $f_n$.

This is often a feature of such "double induction" proofs. Shelah (1987) found a proof with induction only on $m$, giving a bound of $W(m, k) \leq f_4(m+k)$.

Graham offered $1000$ for proof of $W(m) \leq f_5(m) = 2^{2^{2^{2^m}}}$.

Covers (1998) showed $W(m) \leq 2^{2^{2^{2^m}}}$.

The best lower bound known is $W(m) \geq \frac{2^m}{2m}$.

**Corollary 7**

Whenever $\mathbb{N}$ is coloured with finitely many colours, some colour class contains arbitrarily long arithmetic progressions.

What about:

$\mathbb{N}$ finitely coloured $\Rightarrow$ $\exists$ an infinite monochromatic A.P.? This is not true, e.g.

\[ \begin{array}{cccccc}
    & & & & & \\
    & & & & & \\
    & & & & & \\
\end{array} \]

Alternatively, list all infinite A.P.s as $A_1, A_2, \ldots$.

Choose $x_1, y_1 \in A_1$ ($x_1 \neq y_1$), and name $x_1$ red, $y_1$ blue.

Choose new point, $x_2, y_2 \in A_2$ ($x_2 \neq x_1, y_1$), distinct ($x_2 \neq y_1$), and make $x_2$ red, $y_2$ blue. Continue.
Theorem 8 (Strengthened van der Waerden)

Let \( n \in \mathbb{N} \). Then whenever \( \mathbb{N} \) is finitely coloured, there exist an arithmetic progression such that, together with its common difference, is monochromatic.

Proof (by induction on \( k \), the number of colours)

Given a suitable \( n \) for \( k-1 \) (i.e. \( n \) such that whenever \( [n] \) is \( k-1 \)-coloured \( \exists \) a monochromatic AP with common difference of length \( m \)), we will show that \( W(n(m-1)+1, k) \) is suitable for \( k \).

Given a \( k \)-colouring of \( [W(n(m-1)+1, k)] \), we have a monochromatic arithmetic progression of length \( n(m-1)+1 \), say

\[
a, a+d, a+2d, \ldots, a+n(m-1)d
\]

is red.

If \( d \) is red, we are done. Similarly, if \( \exists 1 \leq r < n \)

with \( rd \) red, we are done. (First term \( a \))

So WLOG, \([d, 2d, \ldots, nd]\) is \((k-1)\)-coloured, so we are done by induction.

\[
\left(\frac{a}{d}\right)^3 = \left(\frac{b}{e}\right)^2
\]

Remarks

1. Henceforth, we do not care about bounds.

2. Case \( m=2 \) is Schur's Theorem:

\( WNFC \) (whenever \( \mathbb{N} \) is finitely coloured) \( \exists x, y, z \) monochromatic with \( x+y = z \). This can also be deduced from Ramsey's Theorem directly (exercise).
The Hales - Jewett Theorem

Let $X$ be a finite set. A subset of $X^n$ (the $n$-dimensional cube on alphabet $X$) is a line or combinatorial line if

$\exists I \subseteq [n], I \neq \emptyset \text{ and } a_i \in X, \text{ each } i \in [n] \setminus I \text{ such that}$

$L = \{ x = (x_1, \ldots, x_n) \in X^n : x_i = a_i \text{ for } i \notin I, x_i = x_j \forall i, j \in I \}$

We say that $I$ is the set of 'active coordinates'.

e.g. in $[3]^2$

In $[3]^3$, we could have

$\{ (1,1,1), (2,2,1), (3,3,1) \}, I = \{1,2\}$

$\{ (1,1,1), (2,2,2), (3,3,3) \}, I = \{1,2,3\}$

$\{ (2,3,1), (2,3,2), (2,3,3) \}, I = \{3\}$

Note

The definition is unchanged if we permute $X$.

Theorem 9 (Hales - Jewett Theorem)

$\forall k, m, \exists n \text{ such that whenever } [m]^n \text{ is } k \text{-coloured, there is a}$

$\text{monochromatic line.}$

Note

1. The smallest such $n$ is denoted $HS(m, k)$.

2. So $m$-in-a-row, noughts and crosses, played in enough dimensions, cannot end in a draw. (exercise: player 1 wins)

3. Hales - Jewett & van der Waerden
Indeed, given a $k$-coloring of $\mathbb{N}$, induce a $k$-coloring of $[m]^n$ ($n$ large) by $C'(\langle x_1, \ldots, x_n \rangle) = C(x_1 + \ldots + x_n)$.

We have a monochromatic line for $C'$ ($n$ large enough).

Giving a monochromatic A.P. in $\mathbb{N}$ of length $m$ (common difference = # active coordinates)

For a line $L$ in $[m]^n$, write $L^-$ and $L^+$ for its first and last points (in the ordering $x \leq y$ if $x_i \leq y_i \forall i$).

We say that lines $L_1, \ldots, L_r$ are *focused* at $f$ if $L_i^+ = f \forall i$.

We say that they are *colour-focused* (for a given colouring) if in addition, each $L_i \setminus \{L_i^+\}$ is monochromatic, and no two are the same colour.

E.g. $[4]^2$
\[ \exists I = [n] \setminus \emptyset, a_i \in \mathbb{X} \text{ for } i \in [n] \setminus I \]

such that

\[ L = \{ x = (x_1, \ldots, x_n) \in \mathbb{X}^n : x_i = a_i \forall i \notin I, \quad x_i = x_j \forall i, j \in I \} \]

\( I \) : active coordinates

For \( I = \{i_1, \ldots, i_k\} \), \( k: x_{i_1} = x_{i_2} = \ldots = x_{i_k} \), but the value \( k \) varies over points \( x \).

For \( i \notin I \), \( x_i = a_i \), constant over all points \( x \).
Proof (of Theorem 2)

Induction on $m$. $m = 1$ trivial.

Given $m > 1$, we may assume that $HS(m-1, k)$ exist, $\forall k$.

Claim: $\forall 1 \leq r \leq k \exists n$ such that $[m]^n$ $k$-coloured gives

i) A monochromatic line

or

ii) $r$ colour focussed lines

(Then we are done by setting $r = k$ and looking at the focus)

We prove the claim by induction on $r$. $r = 1$ is done, taking $n = HS(m-1, k)$. Given $n$ suitable for $r-1$, we'll show that $n + HS(m-1, k^n)$ is suitable for $r$.

Write $n'$ for $HS(m-1, k^n)$. Given a $k$-colouring $c$ of $[m]^{n+n'}$, suppose that we have no monochromatic line.

View $[m]^{n+n'}$ as $[m]^n \times [m]^n$. There are $k^{n^n}$ ways to colour a copy of $[m]^n$, so by definition of $n'$, if a line $L$ in $[m]^n$, active coordinates $I$, such that

$$[m]^n \rightarrow \bigoplus_{c'} [m]^n$$

(identically coloured copies of $[m]^n$ (colouring called $c'$))

$\forall a \in [m]^n$. $\forall b, b' \in L \setminus [L^+]$ we have $c((a, b)) = c((a, b')) = c(a)$

say.

By definition of $n'$, we have $r-1$ colour focussed lines for $c'$, say $L_1, \ldots, L_{r-1}$, active coordinate sets $I_1, \ldots, I_{r-1}$ focussed at $f$.

Let $L_i$ be the line in $[m]^{n+n'}$ through $(L_i^-, L_i^+)$, active coordinates $I_i \cup I$. Then $L_i', \ldots, L_i^{r-1}$ are colour focussed at $(f, L_i^+)$. 

Also, the line through \((t, L^-)\) with active coordinate \(I_i\) is monochromatic of a different colour to the \(L_i\).

For \(d \geq 1\), a \(d\)-dimensional subspace or \(d\)-parameter set in \(X^d\) is a subset \(S \subseteq X^d\) such that for some disjoint, non-empty \(I_1, \ldots, I_d \subseteq [n]\) and some \(a_i \in X\), each \(i \in [n] \setminus (I_1 \cup \ldots \cup I_d)\) we have \(S = \{x \in X^d : x_i = a_i \forall i \notin I_1 \cup \ldots \cup I_d, x_i = x_j \forall i, j \in I_k, \forall k\}

e.g. in \([3]^3\):

\[
\begin{cases}
(x, y, 1) : x, y \in [3]^3 \text{ is a 2-parameter set.} \\
(x, y, y) : x, y \in [3]^3
\end{cases}
\]

**Theorem 10** (Extended Hales-Jewett theorem)

\(\forall n, k, d, \exists n \text{ such that } [n]^n \text{ } k\text{-colored } \Rightarrow \exists \text{ a monochromatic } d\text{-parameter set.}\)

(Looks much harder than Hales-Jewett, but...)

**Proof**

View \(X^{dn}\) as \((X^d)^n\), a cube on alphabet \(X^d\).

A line in \((X^d)^n\) (alphabet \(X^d\)) corresponds to a \(d\)-parameter set in \(X^{dn}\) (alphabet \(X\)). Then we are done by taking

\[n = d \cdot HS(m^d, k)\]

Let \(S\) be a finite subset of \(N^d\).

A homothetic copy of \(S\) is a set of the form \(a + kS\) for some \(a \in N^d, k \in N\).

e.g. in \(N\), a homothetic copy of \([1, 2, \ldots, m]\) is an arithmetic progression of length \(m\).
In $\mathbb{N}^2$, a homothetic copy of $[1,2] \times [1,2]$ is a square.

**Theorem II (Gallai's Theorem)**

Let $A$, a finite $S \subseteq \mathbb{N}^d$, whenever $\mathbb{N}^d$ is finitely coloured, we have a monochromatic, homothetic copy of $S$.

**Proof**

Let $S = \{S(1), \ldots, S(m)\}$.

Given a $k$-colouring of $\mathbb{N}^d$, induce a $k$-colouring of $\mathbb{N}^{d'}$ (large) by $c'(x_1, \ldots, x_d) = c(S(x_1) + \ldots + S(x_d))$.

We have a monochromatic line for $c'$ (large), which corresponds to a monochromatic, homothetic copy of $S$ (with $d = \#$ active coordinates).

**Remarks**

1. We can also prove this using Fourier and a product argument.
2. For any $S = [1,2] \times [1,2]$, we applied Hales-Jewett (with $m = 4$). What if we had applied extended Hales-Jewett for 2-parameter sets with $m = 2$? We would obtain just a monochromatic rectangle when looking for a square.
Chapter 2: Partition Regular Equations

Let $A$ be an $m \times n$ matrix with natural entries. We say that $A$ is partition-regular if WFNC, $\exists x \in \mathbb{N}^n$, monochromatic, with $Ax = 0$. (PR denotes 'Partition Regular')

- e.g. $(1, 1, -1)$ is PR: WFNC, $\exists$ monochromatic $x, y, z \in \mathbb{N}$ with $(11-1)(\frac{1}{2})$ i.e. $x + y = z$, Schur's Theorem

- Strengthened van der Waerden says

\[
\begin{pmatrix}
1 & 2 & 1 & \cdots & 1 & 0 \\
0 & 1 & 1 & \cdots & 1 & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 1 & 1
\end{pmatrix}
\]

- $(2, 3, -5)$ is PR; take any $x = y = z$

What about $(2, 3, -6)$?

Remarks

1. $A$ is PR $\iff$ $\lambda A$ is PR (for any $\lambda \in \mathbb{Q} \setminus \{0\}$) so if we wish, we can assume that all entries of $A$ are integers.

2. We can also speak of the 'system of equations $Ax = 0$' being partition regular.

3. Not every matrix is PR e.g. $(2, -1)$ is not PR. Indeed, if it were PR, we could solve $y = 2x$, $x, y$ monochromatic, in any finite colouring, which is clearly false.

For example, colour by whether $\max \{n : 2^n \mid x\}$ is even or odd.

$(2, -1)$ PR $\iff \lambda = 1$

Which matrices are PR?
Let $A$ be an $m \times n$ rational matrix with columns $c^{(i)}$, ..., $c^{(n)}$. 
$A = \begin{pmatrix} c^{(1)} & c^{(2)} & \cdots & c^{(n)} \end{pmatrix}$. Each $c^{(i)} \in \mathbb{Q}^n$.

We say that $A$ has the column property if there exists a partition $[n] = B, u \ldots v B_d$ such that

i) $\sum_{i \in B_0} c^{(i)} = 0$

ii) $\sum_{i \in B_1} c^{(i)} \in \langle c^{(i)} : i \in B, u \ldots v B_{r-1} \rangle$, $2 \leq r \leq d$ where $\langle \ldots \rangle$ denotes linear span (span over $\mathbb{Q}$).

**Examples**

1. $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ has CP (the column property).

2. $\begin{pmatrix} 1 & 2 & -1 & 0 \\ m & 0 & \cdots & \cdots & -1 \end{pmatrix}$ has CP. Take $B_1 = \{1, 3, 4, 5, \ldots, m+2\}$

3. $\begin{pmatrix} 1 & 2 & 3 & -5 \end{pmatrix}$ has CP. $B$, contains everything.

4. $(A - I)$ has CP $\iff \lambda = 1$.

We aim to prove Rado's Theorem:

PR $\iff$ CP

**Notes**

i) This can check if A is PR in finite time.

ii) Neither direction is obvious.

(We expect $\iff$ to be harder.)

We start with Rado for a single equation. We want that if $a_1, \ldots, a_n$ are non-zero rationals, then $(a_1 a_2 \ldots a_n)$ is PR

$\iff \sum_{i \in I} a_i = 0$ for some $I \neq \emptyset$. 


Let $p$ be prime. We colour $\mathbb{Z}$ by giving $x$ the colour $d(x)$, its last non-zero digit in its base $p$ expansion.

For example, if $x = x_n p^n + x_{n-1} p^{n-1} + \ldots + x_0 p + x_{-1}$ $(0 \leq x_i \leq p-1)$ for $i$. We set $L(x) = \min \{ i : x_i = 0 \}$ and $d(x) = x_L(x)$. 

E.g. if $x$ in base $p$ is $3014720070000$ then $L(x) = 4$, $d(x) = 7$.

Proposition 1

Let $a_1, \ldots, a_n \in \mathbb{Q} \setminus \{0\}$. Then $(a_1, \ldots, a_n) \not\approx \sum_{i \in I} a_i = 0$, some $I \neq \emptyset$.

Proof

We may assume that $a_i \in \mathbb{Z}$ for all $i$ (multiplying up if necessary).

Fix a prime $p$, $p > \sum_{i} |a_i|$, and consider the above colouring. We have monochromatic $x_1, \ldots, x_n$ with $\sum_{i} a_i x_i = 0$, and say $d(x_i) = d \forall i$, N.B. $x_i$ can have final digit $d$ in different places.

E.g. $x_1: \ldots \, d \, 0 \, 0 \, 0 \, 0 \, 0 \, 0 \, 0 \\
    x_2: \ldots \, d \, 0 \, 0 \, 0 \, 0 \\
    x_3: \ldots \, 0 \, 0 \, 0 \, 0 \, 0 \\
    x_4: \ldots \, d \, 0 \, 0 \, 0 \, 0 \, 0 \, 0 \, 0 \\
    x_5: \ldots \, d \, 0 \, 0 \, 0 \, 0 \, 0 \, 0 \, 0 \\

Let $L = \min \{ L(x_i) : 1 \leq i \leq n \}$ and let $I = \{ i : L(x_i) = L \}$.

Then considering $\sum_{i \in I} a_i x_i = 0$, computed in base $p$, we have $\sum_{i \in I} d a_i \equiv 0 \pmod{p}$ so $\sum_{i \in I} a_i \equiv 0 \pmod{p}$ (prime). Therefore $\sum_{i \in I} a_i = 0$ (by choice of $p$).

Remarks

1. Or we could have said that for each prime $p$, we have $I$ with $\sum_{i \in I} a_i \equiv 0 \pmod{p}$, some set $I$ is used infinitely often, whence $\sum_{i \in I} a_i = 0$. 

2. We coloured by 'end' mod $p$. We can also colour by 'start' mod $p$, but this is harder.

3. No other ways to prove proposition 1 are known.
For the other direction, we start with the first non-trivial case, namely $(1, 1, 1)$.

**Lemma 2**

Let $L \leq Q$. Then WNFC $\exists$ monochromatic $x, y, z$ with $x + y = z$.

**Proof**

WLOG $L > 0$ (because we can deal with $L = 0$, and we can rewrite for $L < 0$ as $x - ly = x$). Say $L = \frac{r}{s}$ where $r, s \in \mathbb{N}$.

Proceed by induction on $k$, the number of colors. This is trivial for $k = 1$ (taking $x = 1, y = s, z = 1 + r$, and max $(s, 1 + r)$ a suitable $n$). Given $n$ suitable for $k-1$ we show that $\exists W(n+1, k)$ is suitable for $k$. Indeed, given a $k$-colouring of $[\mathbb{S}W(n+1, k)]$, we have a monochromatic AP of length $nr + 1$. Inside $[W(n+1, k)]$, say $a, a + d, \ldots, a + nd$ are all red.

If any of $isd, 1 \leq i \leq r$ are red, we are done: $a + \frac{s}{r}(isd) = a + nd$. So we may assume that $sd, 2sd, 3sd, \ldots, nsd$ is $(k-1)$-coloured, and we are done by induction.

**Remark**

This is very similar to the proof of Strengthened van der Waerden.
**Theorem 3** (Rado for Single Equations)

Let \( a_1, \ldots, a_n \in \mathbb{Q} \setminus \{0\} \). Then \( (a_1, a_2, \ldots, a_n) \) is PR if \( \sum_{i \in I} a_i = 0 \) for some \( I \neq \emptyset \).

**Proof**

(\( \Rightarrow \)) This is proposition 1.

(\( \Leftarrow \)) Fix some \( i_0 \in I \). For suitable \( x, y, z \), we set
\[
\begin{align*}
\mathcal{C}_0 &= x, \\
\mathcal{C}_i &= z & \forall i \in I \setminus \{i_0\}, \\
x_{i_0} &= y & \forall i \in I.
\end{align*}
\]

We want \( \sum a_i x_i = 0 \), all \( x_i \) the same colour (in a given colouring). So we want \( x, y, z \) monochromatic such that
\[
a_{i_0} x + \left( \sum_{i \in I \setminus \{i_0\}} a_i \right) z + \left( \sum_{i \in I} a_i \right) y = 0
\]
i.e.
\[
a_{i_0} x - a_{i_0} z + \left( \sum_{i \in I} a_i \right) y = 0
\]
i.e.
\[
x + \frac{1}{a_{i_0}} \left( \sum_{i \in I} a_i \right) y = z.
\]
Hence we are done by Lemma 2.

**Rado's Boundedness Conjecture**

If \( m \times n \) matrix \( A \) is not PR, then there exists a 'bad' \( k \)-colouring for some \( k \). Is \( k \) bounded (for fixed \( m, n \))?

Equivalently, is there a \( K = K(m, n) \) such that if an \( m \times n \) \( A \) is PR for \( k \) colours then it is PR.

This is known for \( 1 \times 3 \) (Fox, Kleitman, 2006) - 24 colours is enough.

The answer is not known for any other case.

**Proposition 4**

If \( m \times n \) \( A \) is PR then \( A \) has CP
Proof:

WLOG all entries of $A$ are integers. Let $c^{(1)}, \ldots, c^{(n)}$ be the columns of $A$. For a prime $p$, we have a $(p-1)$-colouring of $\mathbb{N}$ (each has colour $d(x)$) so we have monochromatic $x_1, \ldots, x_n$ such that $x_1 c^{(1)} + \ldots + x_n c^{(n)} = 0$, say all $x_i$ have colour $d(x_i)$.

Rightmost for $B_1$: Partition $\mathbb{N}$ as $B_1 \cup \ldots \cup B_r$ and so on, where $B_i$ consists of the $i$ for which $x_i$ is rightmost ending, and so on, as in the diagram. For infinitely many $p$, say all $p \equiv 0 \pmod{p}$, we get the same (ordered) partitions.

Given $p \equiv 0 \pmod{p}$, we have $x_1, \ldots, x_n$ and $d$ and $B_1, \ldots, B_r$ as above, so considering $\sum x_i c^{(i)} = 0$, performed in base $p$,

we have:

1) $\sum_{i \in B_1} d c^{(i)} = 0 \pmod{p}$

2) For each $2 \leq s \leq r$, $p^t \sum_{i \in B_s} d c^{(i)} + \sum x_i c^{(i)} \equiv 0 \pmod{p^{t+1}}$

From 1) we have $\sum_{i \in B_1} c^{(i)} \equiv 0 \pmod{p}$ (divisible mod $p$).

This holds for all $p \equiv 0 \pmod{p}$, so $\sum_{i \in B_1} c^{(i)} = 0$. For $2 \leq s \leq r$, we have $p^t \sum_{i \in B_s} c^{(i)} + \sum_{i \in B_s} (d x_i) c^{(i)} \equiv 0 \pmod{p}$.

Claim:

$\sum_{i \in B_1} c^{(i)} \in \langle c^{(i)} : i \in B_1 \cup \ldots \cup B_{s-1} \rangle$

Proof of Claim:

Suppose not. Then $\exists u \in \mathbb{Z}^n$ such that $u \cdot c^{(i)} = 0 \forall i \in B_1 \cup \ldots \cup B_{s-1}$.
and $u \cdot \sum_{i \in B_S} c^{(i)} \neq 0$. (Thick vector-spaces)

We dot with $u$: $p^T u \cdot \sum_{i \in B_S} c^{(i)} + 0 \equiv 0 (p^{\perp})$

whence $u \cdot \sum_{i \in B_S} c^{(i)} \equiv 0 (p)$. This holds for all $p \in P$, so $u \cdot \sum_{i \in B_S} c^{(i)} = 0$. $\square$
Let $m, p, c \in \mathbb{N}$. A subset $S \subseteq \mathbb{N}$ is an $(m, p, c)$-set on generators $x_1, \ldots, x_m \in \mathbb{N}$ if

$$S = \left\{ \sum_{i=1}^{m} \lambda_i x_i : \exists i \text{ with } \lambda_i = 0 \forall i < j, \lambda_i = c, \lambda_i \in [p, p] \forall i > j \right\}$$

So $S$ is all numbers of the form $c x_1 + \lambda_2 x_2 + \ldots + \lambda_m x_m$ ($\lambda_i \in [p, p] \forall i$)

the rows of $S$ include:

$$\begin{align*}
&c x_2 + \lambda_3 x_3 + \ldots + \lambda_m x_m \quad (\lambda_i \in [p, p] \forall i) \\
&\ldots \\
&c x_m
\end{align*}$$

"Iterated $\mathbb{AP} + \mathbb{CD}$ with $c$ as well" (like $x + \delta y = z$)

e.g. a $(2, p, 1)$ set is $x, -p x_2, x_1 - (p-1) x_2, \ldots, x + p x_2, \text{ and } x$

This is an $\mathbb{AP}$ with $\mathbb{CD}$.

A $(2, p, 3)$-set is $3 x_1 - p x_2, 3 x_1 - (p-1) x_2, \ldots, 3 x_1 + p x_2$ and $3 x_2$

An $\mathbb{AP}$ whose middle term is a multiple of 3, and 3 $\mathbb{CD}$.

Theorem 5

WNFC, there exists a monochromatic $(m, p, c)$-set (asym, $p, c \in \mathbb{N}$). Let $R_i$ will contain $i$th row of $(M, p, c)$ set.

Proof: $B_i$ contains set of good generator so for for rows $1, 2, \ldots, i$

Let $\mathbb{N}$ be $k$-coloured. $A_i$: restrict $B_i$ to multiples of $i$ to continue the process.

Idea: $G$ for an $(m, p, c)$-set, $M = k(m-1) + 1$, with each row monochromatic.

Let $n$ be large (large enough for everything to come).

Let $A_1 = \left\{ c, 2c, \ldots, \frac{n}{c} \right\}$

Inside $A_1$, we have a monochromatic $\mathbb{AP}$

$$R_1 = \left\{ c x_1 - n d, c x_1 - (n-1) d, \ldots, c x_n - n d \right\}, \text{ for large enough } n.$$

Let $\mathbb{R}_i$ be any ray of colour $c_i$.\[\]
Let \( B_1 = \{ d_1, 2d_1, \ldots, \frac{n_1}{pm} d_1 \} \). Note that if \( x_1, \ldots, x_m \in B_1 \) and \( \lambda_1, \ldots, \lambda_m \in [\frac{1}{2}, \frac{1}{2}] \), then \( cx_1 + \lambda_1 x_2 + \cdots + \lambda_m x_m \in \mathbb{R} \), so is colour \( k_1 \).

Let \( A_2 = \{ cd_1, 2cd_1, \ldots, \frac{n_1}{pm} cd_1 \} \). Inside \( A_2 \) we have monochromatic AP \( R_2 = \{ cx_2 - n_2 d_2, cx_2 - (n_2 - 1) d_2, \ldots, cx_2 + n_2 d_2 \} \) with \( n_2 \) large. Say \( R_2 \) is of colour \( k_2 \).

Let \( B_2 = \{ d_2, 2d_2, \ldots, \frac{n_2}{pm} d_2 \} \). Note that if \( x_1, \ldots, x_m \in B_2 \) and \( \lambda_1, \ldots, \lambda_m \in [\frac{1}{2}, \frac{1}{2}] \), then \( cx_2 + \lambda_1 x_3 + \cdots + \lambda_m x_m \in \mathbb{R} \), so is colour \( k_2 \).

Continuing, we obtain an \((m, p, c)\) - set on generators \( x_1, \ldots, x_m \) with each row monochromatic. But then some \( m \) rows are the same colour since \( M = (m-1)k + 1 \), giving a monochromatic \((m, p, c)\) - set.

\[ \Box \]

**Proposition 6**

- **Columns property**

Let \( mx \times n \) \( A \) have \( CP \). Then \( \exists (m, p, c) \) such that every \((m, p, c)\) - set contains a solution to \( Ax = 0 \).

**Idea:** \( CP \) enables solution to be easily constructed.

**Proof**

- Then \( m, p, c \) are chosen.

\[ A = \begin{pmatrix} \uparrow & \uparrow & \uparrow \\ C^{(1)} & C^{(2)} & \cdots & C^{(n)} \\ \downarrow & \downarrow & \downarrow \end{pmatrix} \]

We have a partition \( [n] = B_1 \cup \cdots \cup B_r \),

\[ \sum_{i \in B_s} c^{(i)} = 0 \]

\( (\text{and } \sum_{i \in B_s} c^{(i)} = 0) \). We say that

\[ \sum_{i \in B_s} c^{(i)} = \sum_{i \in B_s, B_{s-1}} q_is c^{(s)} \text{ for some rationals } q_is \]
Define \( \text{dis} = \begin{cases} 0 & \text{if } i \notin B_i \cup \ldots \cup B_s \cup \ldots \cup B_{s-1} \\ 1 & \text{if } i \in B_i \cup \ldots \cup B_s \\ 2 \text{dis} & \text{if } i \in B_i \cup \ldots \cup B_{s-1} \end{cases} \) (and with a 1 for each s).

Hence \( \sum_{i=1}^{n} \text{dis} c^{(i)} = 0 \) (for each \( 1 \leq s \leq r \)).

Given \( x_1, \ldots, x_r \in \mathbb{N} \), put \( y_i = \sum_{s=1}^{r} \text{dis} x_s \) (for \( 1 \leq i \leq n \)).

Then \( \sum_{i} y_i c^{(i)} = \sum_{i} \sum_{s} \text{dis} x_s c^{(i)} \)

want \( = 0 \)

So we are done. Set \( m = r \), \( c = \text{LCM of denominators of dis} \)

\( p = c \max |\text{dis}| \). Then \( y_1, \ldots, y_r \) are all in the \((m, p, c)\)-set on generators \( x_m, x_{m-1}, \ldots, x_1 \).

**Theorem B** (Rado's Theorem)

Let \( A \) be an \( n \times n \) matrix with rational entries. Then

\( A \) \text{ PR } \iff \( A \) has \( CP \).

**Proof**

(\( \Rightarrow \)) Proposition 4.

(\( \Leftarrow \)) Theorem 5 and Proposition 6.
Remark

1. Given Rado, results like Schur or van der Waerden are just trivial.

2. If a matrix A is PR for all last-digit-base-p colorings, then (by the proof of Rado) we know that A is PR for all colorings.

   No direct proof is known.

   For $x_1, \ldots, x_m \in \mathbb{N}$, write $FS(x_1, \ldots, x_m)$ for $\{ \sum_{i \in I} x_i : I \neq \emptyset \}$

The case $(m, 1, 1)$ of Theorem 5 immediately gives:

Theorem 8 (Finite Sums Theorem / Folkman's Theorem / Sanders Theorem)

For $m, WNFC$, $\exists x_1, \ldots, x_m$ with $FS(x_1, \ldots, x_m)$ monochromatic.

Remarks

1. Alternatively, check that the matrix has CP.

2. The case $m = 2$ is Schur.

3. What about finding a monochromatic $FP(x_1, \ldots, x_m) = \{ \sum_{i \in I} x_i : I \neq \emptyset \}$?

   Yes, just look at $\{2^1, 2^2, 2^3, \ldots \}$ and apply the finite sums theorem.

4. What about finding monochromatic $PS(x_1, \ldots, x_m) \cup FP(x_1, \ldots, x_m)$?

   This is unknown.

   The case $m = 2$ would be to find monochromatic $x, y, xy, xy$.

   This is also unknown.

   except $x = y = 2$

   What about finding $x, y$ with $x + y, xy$ the same color?

   Also unknown.
Corollary 9 (Consistency Theorem)

\[ A, B \text{ PR } \rightarrow (A \circ B) \text{ PR} \]

(i.e. if we can always solve \( Ax = 0 \) in one colour class, and \( By = 0 \) in one colour class, then we can solve both in one colour class).

Proof

Trivial by CP.

Remark

This can also be proved directly (i.e. not via Rado) but this is much harder.

More is true.

Corollary 10

WNFC, some colour class contains solutions to all PR matrices.

Proof

Suppose not. Then we have \( N = D_1, v \ldots v D_r \), where for each \( i \) there is a PR matrix \( A_i \) such that \( D_i \) contains no solution to \( A_i x = 0 \).

Let \( A = (A_1, A_2, \ldots, A_r) \). Then \( A \) is PR by Corollary 9, but no \( D_i \) contains a solution to \( A_i x = 0 \).

A set \( D \subset N \) is called partition regular if it contains solutions to all PR matrices. (e.g. \( N \))

So Corollary 10 says that when \( N = D_1, v \ldots v D_r \), then some \( D_i \) is PR.
Rado's Conjecture (1933)

If $D$ is PR, $D = D_0 u ... u D_k$, then some $D_i$ is PR.

This was proved by Denker (1973) via $(m, p, c)$-sets.

Hindman's Theorem

Aim

To show that WNFc, $\exists x_1, x_2, ...$ with $FS(x_1, x_2, ...)$ monochrome

This will be our first infinite PR system.

Filters and Ultrafilters

Roughly "a filter is a notion of which subsets of $\mathbb{N}$ are large" and
"an ultrafilter is a more precise one".

A filter is a non-empty $F \subseteq \mathcal{P}(\mathbb{N})$ such that

i) $\emptyset \notin F$

ii) $A \in F$, $B \supseteq A \Rightarrow B \in F$ ("$F$ is an up-set")

iii) $A, B \in F \Rightarrow A \cap B \in F$ ("$F$ closed under finite intersections")

Examples

1. $\{A \subseteq \mathbb{N} : 4 \in A\}$

2. $\{A \subseteq \mathbb{N} : 4, 5 \in A\}$

3. Non-example $\{A \subseteq \mathbb{N} : |A| = 00\}$, since $\text{Odds} \cap \text{Evens} = \emptyset$

4. $\{A \subseteq \mathbb{N} : A^c \text{ finite}\}$, the cofinite filter

5. $\{A \subseteq \mathbb{N} : \text{Evens} \setminus A \text{ finite}\}$

An ultrafilter is a maximal filter (we can't add any more sets).

Of the above,
1. If maximal, indeed for any \( n \in \mathbb{N}\) we have \( \bar{n} = \{ A \subseteq \mathbb{N} : n \in A \} \),

   called the "principle ultrafilter at \( n \)".

2. No, as I explain extends it.

4. No, as 5 extends it.

5. No, as we replace Evens with \( \{ n : n \text{ a multiple of } 4 \} \).

Proposition 11

A filter \( F \) is an ultrafilter \( \iff \forall A, A \subseteq F \text{ or } A^c \subseteq F \).

Proof:

\( \leq \) We cannot add any new \( A \in F \) as \( A^c \) is already in \( F \),

whence \( A \cap A^c = \emptyset \in F \).

\( \geq \) Given \( A \notin F \), we must have \( B \cap A = \emptyset \) for some \( B \in F \)

otherwise, we could extend \( F \) to \( \{ D \subseteq \mathbb{N} : D \geq A \cap B, \text{ some } B \in F \} \).

So \( B \supseteq A^c \), so \( A^c \in F \).
Remain
Similarly, if \( U \) is an ultrafilter, \( A \in U \), \( A = B \cup C \), then \( B \in U \) or \( C \in U \). Indeed, if not then \( B^c, C^c \in U \), where \( A^c = B^c \cap C^c \in U \). **

Theorem 12
Every filter is contained in an ultrafilter.

Note
Any ultrafilter extending the cofinite filter is non-principal.
Conversely, if ultrafilter \( U \) is non-principal, then it extends the cofinite filter, since if finite \( A \in U \) exists, then applying the remark above (repeatedly), we would get \( J \in U \), some \( n \).

Proof
Given a filter \( F \), we seek a maximal filter \( F > F \). So, by Zorn's Lemma, it is enough to show that any non-empty chain \( \{ F_i : i \in I \} \) has an upper bound.

Put \( F = \bigcup_{i \in I} F_i \).

Then \( F > F_i \) \( \forall i \), so we just need to check that \( F \) is a filter.

i) \( \phi \notin F \) since \( \forall i \in I, \phi \notin F_i \)

ii) \( \text{Given } A \in F, B \supseteq A \), we have \( A \in F_i \) for some \( i \), so \( B \in F_i \), so \( B \in F \).

iii) \( \text{Given } A, B \in F \), we have \( A \in F_i \), \( B \in F_i \), for some \( i, j \).
WLOG $F_i \supseteq F_j$; since we have a chain.

Then $A_i \cap B_j \subseteq A \cap B \subseteq F_i \subseteq B \subseteq F_j$.

\[ \Box \]

**Remark**

We do need some form of the Axiom of Choice to get non-principal ultrafilters.

The set of all ultrafilters is denoted $\beta N$. We can put a topology on $\beta N$, given by a base of open sets:

\[ C_A = \{ U \in \beta N : A \in U \}, \quad A \subseteq N. \]

This is a base:

i) $U_{A \subseteq N} \cap C_A = \beta N$

ii) $C_A \cap C_B = C_{A \cap B}$ (since $A, B \in U$ $\iff$ $A \cap B \in U$).

Then the open sets are all sets of the form $\bigcap_{i \in I} C_{A_i} = \{ U : A_i \in U, \forall i \}$.

Basic closed sets are the $C_A$ (because $(C_A)^c = C_{A^c}$ since $A \notin U$ $\iff$ $A^c \in U$). So the closed sets are of the form

\[ \bigcup_{i \in I} C_{A_i} = \{ U : A_i \in U, \forall i \} \]

Each principal $\tilde{n}$ is isolated. Indeed, $C_{\{n\}} = \{ \tilde{n} \}$.

Also, the $\tilde{n}$, for $n \in N$, are dense in $\beta N$. Indeed, $\tilde{n} \in C_A$ $\iff$ $n \in A$. Then, we can view $N$ as a subset of $\beta N$ by identifying $n \in N$ with $\tilde{n} \in \beta N$.

**Theorem 13** $\beta N$ is compact Hausdorff

First, we show that $\beta N$ is Hausdorff. Given distinct $U, V$ \[ F \in \mathcal{N} \text{ with } A \subseteq U, \quad A \notin V. \] Then $U \in C_{A}, \quad V \in C_{A^c}$
For compactness, given closed sets $F_i$, $i \in I$, with the finite intersection property (any finite intersection $F_{i_1} \cap \ldots \cap F_{i_n} \neq \emptyset$), we must show that $\bigcap_{i \in I} F_i \neq \emptyset$.

WLOG, each $F_i$ is basic, say $F_i = C_{a_i}$.

Hence the sets $A_i$, $i \in I$, have the finite intersection property.

Indeed, $C_{a_{i_1}} \cap \ldots \cap C_{a_{i_n}} = C_{a_{i_1} \cap \ldots \cap a_{i_n}} \neq \emptyset$, whence $A_{i_1} \cap \ldots \cap A_{i_n} \neq \emptyset$.

Hence the $A_i$, $i \in I$, generate a filter:

$F = \{ B \subseteq N : B \supseteq A_{i_1} \cap \ldots \cap A_{i_n}, i_1, \ldots, i_n \in I \}$.

Let ultrafilter $U$, extend $F$. Then, $\forall i, A_i \in F \subseteq U_i$.

i.e. $U_i \in C_{a_i}$. \( \square \)

Remark

$\beta N$ is actually the largest compact Hausdorff space in which $N$ is dense ("the largest compactification of $N$"). More precisely,

\[
N \hookrightarrow \beta N \quad \text{given} \quad f : N \to X, \text{any compact Hausdorff space, there exists } g : \beta N \to X \text{ such that } f = g |_N\]

Ultrafilter Quantifiers

For an ultrafilter $U_i$, and a property $p(x)$ ($x \in N$), write

\[
\forall x \in N \quad p(x) \quad \text{if} \quad (x \in N : p(x)) \subseteq U_i.
\]

"For $U$-most $x$, $p(x)$".

e.g. If $U_i = N$ then $\forall x \in N \quad p(x) \iff p(n)$.

For any non-principal $U_i$, $\forall x : x > 10 \iff \{x \in N : x > 10\} \in U_i$. \( \square \)
Warning

∀x x and ∀x x don't commute, even if U = V.
For example, let U be non-principal.
Then ∀x x ∀x y x < y, indeed "∀x y x < y" holds
for all x ∈ N.
But ∀y y ∀x x < y is false.
Indeed "∀x x < y" holds for no y ∈ N.
Proposition 14

Let \( \mathcal{U} \) be an ultrafilter, \( p, q \) statements. Then

i) \( (\forall x)(p(x) \text{ and } q(x)) \iff (\forall x)p(x) \text{ and } (\forall x)q(x) \)

ii) \( (\forall x)(p(x) \text{ or } q(x)) \iff (\forall x)p(x) \text{ or } (\forall x)q(x) \)

iii) \( (\forall x)p(x) \) false \( \iff (\forall x)(p(x) \) false)

Proof

Let \( A = \{ x \in \mathbb{N} : p(x) \} \), \( B = \{ x \in \mathbb{N} : q(x) \} \)

i) \( A \cap B \in \mathcal{U} \iff A \in \mathcal{U} \text{ and } B \in \mathcal{U} \)

ii) \( A \cup B \in \mathcal{U} \iff A \in \mathcal{U} \text{ or } B \in \mathcal{U} \)

iii) \( A \notin \mathcal{U} \iff A^c \in \mathcal{U} \)

\[ \square \]

Definition

\( \mathcal{U} + \mathcal{V} := \{ A : (\forall x)(\forall y)(x + y \in A) \} \)

e.g. \( \tilde{n} + \tilde{m} = (\tilde{n} + \tilde{m}) \)

or without quantifiers

\( \mathcal{U} + \mathcal{V} = \{ A : \{ x : \{ y : x + y \in A \} \in \mathcal{V} \} \in \mathcal{U} \} \)

Note that \( \mathcal{U} + \mathcal{V} \) is an ultrafilter:

- \( \emptyset \notin \mathcal{U} + \mathcal{V} \).
- If \( A \in \mathcal{U} + \mathcal{V} \) and \( B \supset A \) then \( B \in \mathcal{U} + \mathcal{V} \).
- If \( A, B \in \mathcal{U} + \mathcal{V} \) then \( (\forall x)(\forall y)(x + y \in A) \) AND \( (\forall x)(\forall y)(x + y \in B) \)

So \( (\forall x)(\forall y)(x + y \in A \text{ and } x + y \in B) \) (Prop 14, twice)

i.e. \( (\forall x)(\forall y)(x + y \in A \cap B) \)
If $A \neq U + V$ then $\neg (\forall x)(\forall y)(x+y \in A)$

$\Rightarrow (\forall x)(\forall y)(\neg(x+y \in A))$  (Prop 14 twice)

i.e. $(\forall x)(\forall y)(x+y \in A^c)$

We have that + is also associative:

$(U + V) + W = \{ A \subset \mathbb{N} : (\forall x)(\forall y)(\forall z)(x+y+z \in A) \}$

$= U + (V + W)$

Also, + is left continuous i.e. for fixed $V$, the mapping

$U \mapsto U + V$, $\beta\mathbb{N}$ to $\beta\mathbb{N}$, is continuous.

Indeed, given a basic open set $C_A$

$U + V \subseteq C_A \iff A \subseteq U + V$

$\iff \{ x : (\forall y)(x+y \in A) \} \subseteq U$

$\iff U \subseteq C \{ x : (\forall y)(x+y \in A) \}$

**Remark**

In fact, + is not commutative or right continuous.

The key to Hindman will be

**Lemma 15 (Idempotent Lemma)**

$\exists U \in \beta\mathbb{N}$ with $U + U = U$

**Note**  

Stone-Čech compactification of $\mathbb{N}$

All we will use about $\beta\mathbb{N}$ is compactness, Hausdorff, non-emptiness, and that + is associative and left continuous.

**Proof**

Set of all possible $x+y, x,y \in M$

Idea: We go for a minimal $M \subseteq \beta\mathbb{N}$ with $M + M \subseteq M$ and hope that $M = \{ x \}$ for some $x$. 


There exists a compact, non-empty $M \subset \beta \mathbb{N}$ with $M + M \subset M$ (e.g., $M = \beta \mathbb{N}$) and we seek a minimal such $M$.

By Zorn, it is enough to check that if $\{M_i : i \in I\}$ is a chain of such sets then so is $M = \bigcap_{i \in I} M_i$.

i) $M$ is non-empty, because the $M_i$ are closed sets with the Finite Intersection Property (remember that compact $\Rightarrow$ closed in a compact Hausdorff space).

ii) $M$ is an intersection of closed sets, so is closed.

iii) $\forall x, y \in M : x + y \in M_i$ $\forall i$, so $x + y \in M_i$ $\forall i$, so $x + y \in M$.

Let $M$ be a minimal such set. Fix $x \in M$, and we will show that $x + x = x$.

Claim: $M + x = M$

Proof of Claim: We have $M + x \subset M$. (Since $M + M \subset M$)

Also, $M + x \neq \emptyset$ (so $M \neq \emptyset$).

$M + x$ is compact (a continuous image of compact set $M$ under $+$)

$(M + x) + (M + x) = (M + x + M) + x \subset M + x$

Hence $M + x = M$ by minimality of $M$.

So $\exists y \in M$ with $y + x = x$.

Now, let $N = \{y \in M : y + x = x\}$

Claim: $N = M$ (then we are done as $x \in N = \Rightarrow x + x = x$)

Proof of Claim: We have $N \subset M$, by definition.

Also, $N \neq \emptyset$ (by the above, $y \in N$)
It is clear that the inverse image of \( \{x \mid 3 \text{ under continuous map } + \} \) is closed.

Also, \( y, z \in N \Rightarrow (y+z)+x = y+(z+x) = y+x = x \)

so \( y+z \in N \). \( \therefore \ N + N \subseteq N \)

Hence \( N = M \), by minimality of \( M \).

Remarks

1. Hence \( M = \{x \mid 3 \} \) by minimality.

2. Does \( BN \) have any finite (non-trivial) subgroups?

   e.g. \( U \) with \( U + U \neq U \) but \( U + U + U = U \)

   This is the Finite Subgroup Problem. The answer is no [Zelenskii, 1996]

3. Can one ultrafilter absorb another?

   i.e. can we have \( U, V \) with \( U + U, U + V, V + U, V + V = U \)

   This is called the Continuous Homomorphism Problem - unknown.
Theorem 16 (Hindman's Theorem)

\[ \text{WINFC, } \exists x_1, x_2, \ldots \text{ with } FS(x_1, x_2, \ldots) \text{ monochromatic.} \]

Remark

\( U \) is doing "lots of passes and choosing " for us.

Proof

Let \( U \) be an idempotent ultrafilter. We have \( A \in U \), for some colour class \( A \). We'll find \( FS(x_1, x_2, \ldots) \) in \( A \).

\((\forall x, y) (y \in A)\).

So \((\forall x) (\forall y) (x + y \in A)\), since \( U \cup U = U \).

So \((\forall x) (\forall y) (FS(x, y) \subseteq A)\) by Proposition 14.

Choose \( x_1 \) such that \((\forall y) (FS(x_1, y) \subseteq A)\).

(possible since we have a \( U \)-big set of such \( x_1 \))

Inductively, suppose we have chosen \( x_1, \ldots, x_n \) such that

\((\forall y) (FS(x_1, \ldots, x_n, y) \subseteq A)\)

For each \( z \in FS(x_1, \ldots, x_n) \) we have \((\forall y) (z + y \in A)\).

So \((\forall x) (\forall y) (x + y + z \in A)\), since \( U \cup U = U \).

Thus \((\forall x) (\forall y) (FS(x_1, \ldots, x_n, x, y) \subseteq A)\) by Proposition 14.

Choose \( x_{n+1} \) such that \((\forall y) (FS(x_1, \ldots, x_n, y) \subseteq A)\) \( \square \)

Remarks

1. Very few infinite PR systems are known. No "\( \iff \)" characterization is known.

2. An example is the Milliken-Taylor Theorem: \( \text{WINFC} \)

\[ \exists x_1, x_2, \ldots \text{ such that } FS_{1,2} (x_1, x_2, \ldots) \text{ is monochromatic.} \]

Here \( FS_{1,2} (x_1, x_2, \ldots) = \left\{ \frac{\sum_{i \in I} x_i + \sum_{j \in J} 2x_j}{\max I < \min J} : I, J \text{ finite, non-empty} \right\} \)
Similarly for \( FS_{1,3,7,11}(x_1, x_2, \ldots) \) etc.

Sadly, the Consistency Theorem fails for infinite PR systems. It was proved in 1995 that Hindman and Milliken-Taylor\(^{1,2}\) are inconsistent. Hence, there is no "universal" PR system.

Chapter 3: Infinite Ramsey Theory

We know that for any \( r = 1, 2, 3, \ldots \), whenever \( \mathbb{N}^{(r)} \) is 2-coloured, there exists an infinite monochromatic set. What if we coloured the infinite subsets of \( \mathbb{N} \)?

For any infinite set \( M \subseteq \mathbb{N} \), write \( M^{(\omega)} = \{ L \subseteq M : L \text{ infinite} \} \)

So, if we 2-colour \( \mathbb{N}^{(\omega)} \), must there exist a monochromatic \( M \subseteq \mathbb{N}^{(\omega)} \) (i.e. \( M^{(\omega)} \) is all one colour).

E.g. 2-colour \( \mathbb{N}^{(\omega)} \) by giving \( M \) colour red if \( \sum_{x \in M} \frac{1}{x} \) is convergent and blue if \( \sum_{x \in M} \frac{1}{x} \) is divergent.

We could take \( M = \{ 2^n : n = 0, 1, 2, \ldots \} \)

Proposition

There is a 2-colouring of \( \mathbb{N}^{(\omega)} \) with no infinite monochromatic set.

Proof

We seek a 2-colouring \( c \) such that \( \forall M \subseteq \mathbb{N}^{(\omega)}, \forall x \in M, c(M \setminus \{ x \}) \neq c(M) \).

Notice that \( c(M \setminus \{ x, y \}) = c(M) \) and \( c(M \cup \{ x \}) \neq c(M) \).

Define a relation \( \sim \) on \( \mathbb{N}^{(\omega)} \) by:

\[ L \sim M \text{ if } |L \Delta M| < \infty \]
This is clearly an equivalence relation. Let the equivalence classes be the $E_i : i \in I$.

In each class $E_i$, fix an element $M_i$. Colour $\mathbb{N}^{(\omega)}$ by:

For each $M \in \mathbb{N}^{(\omega)}$ we have a unique $M_i$ with $M \sim M_i$.

Colour $M$ red if $1M \Delta M_i$ is even and blue if $1M \Delta M_i$ is odd.

Remark

We do need some form of the Axiom of Choice.

A 2-colouring of $\mathbb{N}^{(\omega)}$ corresponds to a partition $Y \cup Y^c$ of $\mathbb{N}^{(\omega)}$.

We say that $Y$ is Ramsey if $\exists M \in \mathbb{N}^{(\omega)}$ with $M^{(\omega)} \subseteq Y$ or $M^{(\omega)} \subseteq Y^c$. i.e. $M$ is a monochromatic subset.

So Proposition 1 says that not all sets are Ramsey.

But are "nice" sets Ramsey?

We have a metric on $\mathbb{N}^{(\omega)}$:

$$d(L, M) = \begin{cases} 0 & \text{if } L = M \\ \frac{1}{\min(L,M)} & \text{if } L \neq M \end{cases}$$

Equivalently, we have $\mathbb{N}^{(\omega)} \subseteq 2(\mathbb{N}) \leftrightarrow \{0, 1\}^{\omega}$ which has product topology. So a basic neighbourhood of a point $M \in \mathbb{N}^{(\omega)}$ is $\{L \in \mathbb{N}^{(\omega)} : L \cap [n] = M \cap [n]\}$, $n = 1, 2, \ldots$

Equivalently, the basic open sets are, for each finite $A \subseteq \mathbb{N}$, the set $\{M \in \mathbb{N}^{(\omega)} : A \text{ is an initial segment of } M\}$.

This is called the product or usual or $\mathcal{I}$ topology.

Our first aim is to show that open sets are Ramsey.
Write $N^{(\omega)} = \{ A \subseteq N : A \text{ finite} \}$

For $M \in N^{(\omega)}$, $A \in N^{(\omega)}$, write

$$(A, M)^{(\omega)} = \{ L \subseteq N^{(\omega)} : A \text{ is an initial segment of } L \text{ and } L \upharpoonright A \subseteq M \}$$

"Start as $A$; carry on in $M$".

Fix $Y \subseteq N^{(\omega)}$. We say that $M$ accepts $A$ (into $Y$) if $(A, M)^{(\omega)} \subseteq Y$.

We say that $M$ rejects $A$ if no $L \subseteq M^{(\omega)}$ accepts $A$ (into $Y$).

Notes:

1. $M$ need not accept or reject $A$.
2. If $M$ accepts $A$ then every $L \subseteq M^{(\omega)}$ also accepts $A$.
3. If $M$ rejects $A$ then every $L \subseteq M^{(\omega)}$ also rejects $A$.
4. If $M$ accepts $A$, then $M$ also accepts $A \cup B$, for any $B \subseteq M^{(\omega)}$ with $\min B > \max A$.

$$(\emptyset, M)^{(\omega)} = M^{(\omega)}$$

* Gaussian–Priby Explanation *

(starred part)

Want to reject a finite subset of $\{a_1, a_2, \ldots \}$ e.g. $A = \{a_3, a_7, a_{10} \}$

$M^{(\omega)} \supseteq \{a_{11}, a_{12}, \ldots \}$

$M_{n+1}$ rejects $A$ $\Rightarrow$ $\{a_{n+1}, a_{n+2}, \ldots \}$ rejects $A$

$\Rightarrow \{a_1, a_2, \ldots \}$ rejects $A$

(none $a_i, \ldots, a_n$ are not considered in $(A, M)^{(\omega)}$ type sets)
Lemma 2 (Gaviot Pintey Lemma)

Fix $Y \subseteq \mathbb{N}^{\omega}$. Then $\exists M \in \mathbb{N}^{\omega}$ such that either $M$ accepts $\phi$ or $M$ rejects all of its finite subsets.

Proof

* Suppose that no $M \in \mathbb{N}^{\omega}$ accepts $\phi$, i.e., $M$ rejects $\phi$. We will find $a_1 < a_2 < \ldots$ in $\mathbb{N}$ and $M_1 > M_2 > \ldots$ with $a_n \in M_n \forall n$ and $M_n$ rejects all subsets of $\{a_1, \ldots, a_{n-1}\} \forall n$. Then we are done since $\{a_1, a_2, \ldots\}$ rejects all of its finite subsets.

Put $M_1 = \mathbb{N}$, so $M_1$ rejects $\phi$. Bottom case

Having chosen $M_1, \ldots, M_k$ and $a_1, \ldots, a_{k-1}$ suitably, we seek $a_k \in M_k$, $a_k > a_{k-1}$, and $M_{k+1} \subseteq M_k$ such that $M_{k+1}$ rejects all subsets of $\{a_1, \ldots, a_k\}$.

(This is automatic for all subsets of $\{a_1, \ldots, a_{k-1}\}$ since $M_{k+1} \subseteq M_k$)

Let $b_1 \in M_k$, $b_1 > a_{k-1}$. We cannot put $a_k = b_1$, $M_{k+1} = M_k$, so $M_k$ fails to reject some subset of $\{a_1, \ldots, a_{k-1}, b_1\}$, say $t \cup \{b_1\}$, where $b_1 \notin \{a_1, \ldots, a_{k-1}\}$.

Thus some $N_1 \in \mathbb{N}^{\omega}$ accepts $t \cup \{b_1\}$.

Choose $b_2 \in N_1$, $b_2 > b_1$. We cannot put $a_k = b_2$, $M_{k+1} = M_k$, so $M_k$ fails to reject some subset of $\{a_1, \ldots, a_{k-1}, b_2\}$, say $N_2 \in \mathbb{N}^{\omega}$ accepts $t \cup \{b_2\}$, for some $b_2 \notin \{a_1, \ldots, a_{k-1}\}$. Continue.

We obtain $M_k > N_1 > N_2 > \ldots$ and $b_1 < b_2 < \ldots (b_n \in M_k, b_n \in N_{n+1} \forall n \geq 2)$ and
E_1, E_2, \ldots \subseteq \{a_1, \ldots, a_k\} such that N_\alpha accepts E_n \cup \{b_1, b_2\}.

WLOG \ E_n = E \cap N (passing to a subsequence), for some \ E \subseteq \{a_1, \ldots, a_k\}. So \ b_1, b_2, \ldots \ accepts \ E, contradicting \ M \ rejecting \ E \times \ E.

\[ \square \]

**Theorem 3**

Let \( Y \subseteq N^{(\omega)} \) be open. Then \( Y \) is Ramsey.

**Proof**

Choose \( M \) as given by Gowers-Path. If \( M \) accepts \( \phi \):

we have \( M^{(\omega)} \subseteq Y \). \no\ done.

If \( M \) rejects all of its finite subsets:

We must have \( M^{(\omega)} \subseteq Y \). Indeed, suppose some \( L \subseteq M^{(\omega)} \) has \( L \subseteq Y \).

Since \( Y \) is open, some neighborhood of \( L \) is contained in \( Y \).

So \( J \) an initial segment \( A \) of \( L \) with \( (A, M)^{(\omega)} \subseteq Y \).

So certainly \( (A, L)^{(\omega)} \subseteq Y \), contradicting \( M \) rejecting \( A \times L \).

**Remark**

Since \( Y \) is Ramsey \( \iff \ Y \subseteq Y \) Ramsey, we now have all closed sets Ramsey.

**Definition**

The \( * \) or Ellentuck or Mathias topology on \( N^{(\omega)} \) has basic open sets \( (A, M)^{(\omega)} \) with \( A \subseteq M \subseteq N^{(\omega)} \).

This is a base: \( (A, M)^{(\omega)} \cap (A', M')^{(\omega)} = \emptyset \) or \( (A \cup A', M \cup M')^{(\omega)} \).
This is stronger than $\mathcal{I}$ i.e. we have more open sets.

Is it true that $\mathcal{I}$ has basic open sets $(A, N)^{\omega}$ and $(A, L)^{\omega}$?

**Theorem 3'**

Let $Y \subseteq N^{\omega}$ be *open*. Then $Y$ is Ramsey.

**Proof**

The same as Theorem 3, removing the 'overkill'.

**Definition**

We say that $Y \subseteq N^{\omega}$ is completely Ramsey if $\forall A \in N^{(\omega)}$ and $M \in N^{(\omega)}$, $\exists L \subseteq M^{(\omega)}$ with $(A, L)^{\omega} \subseteq Y$ or $Y^c$.

Not all Ramsey sets are completely Ramsey. For example, take the non-Ramsey $\mathcal{I}$ from Proposition 1, and let $Y'$ be:

$$Y' = Y \cup \{M \in N^{(\omega)} : 1 \notin M\}$$

Then $Y'$ is Ramsey: $\{2, 3, 4, \ldots \}^{(\omega)} \subseteq Y'$, but $Y'$ is not completely Ramsey since there is no $M$ with $(\{1, 3, M\})^{(\omega)} \subseteq Y'$ or $Y'$.

**Theorem 4**

If $Y$ is *open*, then $Y$ is completely Ramsey.

**Proof**

Given $A \in N^{(\omega)}$, $M \in N^{(\omega)}$, we seek $L \subseteq M$ such that $(A, L)^{\omega} \subseteq Y$ or $Y^c$.

We "view $(A, M)^{\omega}$ as a copy of $N^{(\omega)}".

Let $M = \{m_1, m_2, \ldots \}$ where $m_1 < m_2 < \ldots$ and WLOG $m_1 > \max A$. 

\[ A \mid m_1, m_2, \ldots \]

We define $f : N^{(\omega)} \to (A, M)^{(\omega)}$, $N \mapsto A \cup \{\ell_i : \ell_i \in N\}$

This is clearly a homeomorphism in the $\ast$ topology.

Let $Y' = \{N \in N^{(\omega)} : f(N) \subseteq Y\} = \{f^{-1}(Y) \}

Then $Y'$ is $\ast$-open (as $Y$ is $\ast$-open).

So $\exists L \in N^{(\omega)}$ with $L^{(\omega)} \subseteq Y'$ or $Y' \subseteq (Y' \text{ Ramsey})$

i.e. $(A, f(L))^{(\omega)} \subseteq Y$ or $Y^c$
A subset $Y$ of a topological space $X$ is nowhere dense if $Y$ is not dense on any open set - i.e. the closure $\bar{Y}$ has empty interior.

i.e. $\forall$ open $O \neq \emptyset$, $\exists O' \subset O$, $O' \neq \emptyset$, with $O' \cap Y = \emptyset$.

E.g. in $\mathbb{R}$: $\left\{ \frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \ldots \right\}$ or $\left\{ \frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \ldots \right\} \cup \{0\}$, but not $\mathbb{Q} \cap (0,1)$

Proposition 5

Every open set has a neighborhood not meeting $Y$

Let $Y \subset N^{(\omega)}$. Then

$Y$ is nowhere dense $\iff \forall (A,M)^{(\omega)}$, $\exists L \in M$ with $(A,L)^{(\omega)} \subset Y^c$.

So that $Y$ is completely Ramsey

Note

The "$\iff$" is a good sign that the topology and combinatorics are meshing nicely.

Proof

The RHS says: every $(A,M)^{(\omega)}$ contains an $(A,L)^{(\omega)} \subset Y^c$.

The LHS says: every $(A,M)^{(\omega)}$ contains a $(B,L)^{(\omega)} \subset Y^c$.

($\Rightarrow$) is now trivial.

($\Leftarrow$) We know that $Y$ is completely Ramsey (since $Y$ is closed).

So $(A,M)^{(\omega)}$ contains $(A,L)^{(\omega)} \subset Y^c$ or $Y^c$.

Hence $(A,L)^{(\omega)} \subset Y^c$ (because $Y$ has no interior).

So $(A,L)^{(\omega)} \subset Y^c$ because $Y^c \supset (\overline{Y})^c$.

We say that $Y \subset X$ is meagre or of first category if it is a countable union of nowhere dense sets.

E.g. in $\mathbb{R}$, $\mathbb{Q}$ is meagre.

Think of 'meagre' as quite small.
e.g. Baire Category: \( X \) a non-empty complete metric space means that \( X \) itself is not meagre in \( X \).

**Theorem 6**

Let \( Y \subset N^{(\omega)} \) be meagre. Then \( \forall (A, M)^{(\omega)} \exists L \subset M \text{ with} \ (A, L)^{(\omega)} \subset Y^c \) (so \( Y \) is completely Ramsey). In particular, \( Y \) meagre. For every open set \( O \), we have open \( O \subset O', O' \subset Y^c \).

i.e we can "remove \( Y \) from \( O \)."

**Proof**

We have \( Y = \bigcup_{n=1}^{\infty} Y_n \) with each \( Y_n \) \(*\)-nowhere dense.

Given \( (A, M)^{(\omega)} \), we have \( M_1 \subset M \text{ with} \ (A, M_1)^{(\omega)} \subset Y^c \) (Proposition 5).

Choose \( x_1 \in M_1 \), \( x_1 > \max A \). By Proposition 5 twice, we get \( M_2 \subset M_1 \text{ with} \ (A, M_2)^{(\omega)} \subset Y^c \) and then \( M_2 \subset M_2' \text{ with} \ (A \cup \{x_1\}, M_2')^{(\omega)} \subset Y^c \). By Proposition 5 four times, we get \( M_3 \subset M_2 \text{ with} \ (A, M_3)^{(\omega)} \subset (A \cup \{x_1\}, M_3)^{(\omega)}, (A \cup \{x_1, x_2\}, M_3)^{(\omega)} \subset (A \cup \{x_2, M_3)^{(\omega)} \)."

Continuing, we obtain \( M_1 \supset M_2 \supset \ldots \) and \( x_1 < x_2 < \ldots \)

with \( x_n \in M_n \forall n \), and \( (A \cup \{x_1, x_2, \ldots\}, M_n)^{(\omega)} \subset Y^c \), \( \forall F \subset \{x_1, \ldots, x_{n-1}\} \).

So \( (A \cup \{x_1, x_2, \ldots\})^{(\omega)} \subset Y^c \forall n, x_n \subset Y^c \).

We say that \( Y \subset X \) is a Baire set or has the property of Baire if \( Y = \emptyset \Delta M \) for some open \( O \), meagre \( M \).

"\( Y \) is nearly an open set."

**Examples**

1. \( (0, 1) \setminus \{0, \} \subset \mathbb{R} \). \( \mathbb{Q} \) is meagre: write as union of individual points.

2. \( (0, 1) \setminus \mathbb{Q} \subset \mathbb{R} \). \( \mathbb{Q} \cap \{0, 1\} \subset \mathbb{R} \text{ meagre} \).
2. Any open \( M = \emptyset \)

3. Any closed \( Y \), we have \( Y = \text{Interior}(Y) \Delta (Y \setminus \text{Interior}(Y)) \)
   \[ \text{Interior}(Y) = (O^{\circ})^c \] "biggest open set inside \( Y \)"
   (\( \text{Interior}(Y) \) contains no non-empty open set)

4. The Baire sets form a \( \sigma \)-Algebra (closed under complement, and countable unions. Indeed
   \[ Y \text{ Baire} \Rightarrow Y = O \Delta M \quad (\text{open, } M \text{ meagre}) \]
   \[ \Rightarrow Y^c = O^c \Delta M = (O' \Delta M') \Delta M = O' \Delta (M' \Delta M) \]
   \[ Y_1, Y_2, ... \text{ Baire} \Rightarrow Y_n = O_n \Delta M_n \quad (\text{open, } M_n \text{ meagre}) \]
   \[ \Rightarrow U_{n=1}^{\infty} Y_n = U_{n=1}^{\infty} O_n \Delta M_n \quad \text{for some } M \subseteq U_{n=1}^{\infty} M_n \text{ so that } M \text{ is meagre.} \]

So Baire is a bit like measurable.

**Theorem 7**

Let \( Y \subseteq N^{(\omega)} \). Then \( Y \) is completely Ramsey

\[ \Rightarrow Y = * \text{ Baire.} \]

**Notes**

i) Hence any \( \mathcal{C} \)-Borel set (Borel meaning in the \( \sigma \)-algebra generated by the open sets) is Ramsey:

\[ Y \text{ \( \mathcal{C} \)-Borel} \Rightarrow Y = * \text{-Borel} \Rightarrow Y = * \text{-Baire} \Rightarrow Y \text{ completely Ramsey} \]

\[ \Rightarrow Y \text{ Ramsey.} \]

ii) Any set that we 'write down' will invariably (nearly always) be Borel.

iii) e.g. \( \exists \text{ finite } M \text{ such that all } \infty \subseteq M, \text{ we have } \sum_{n=1}^{\infty} \frac{1}{1^n} \text{ having} \)
infinitely many 7s in its decimal expansion.

OR finitely many 7s. (easy to check that the colouring is Borel)

**Proof**

(\(\leq\)) We have \(Y = V \Delta Z\), \(V\) *open*, \(Z\) *meagre*.

Given \((A,M)^{(\omega)}\), \(\exists L \subseteq M\) with \((A,L)^{(\omega)} \subseteq W \cup W^c\), and

\(\exists N \subseteq L\) with \((A,N)^{(\omega)} \subseteq Z^c\). Hence either \((A,N)^{(\omega)} \subseteq Z^c \cap W^c \subseteq Y^c\)

or \((A,N)^{(\omega)} \subseteq Z^c \cap W^c \subseteq Y^c\) because \(Y = W \Delta Z\)

(\(\geq\)) We have \(Y = \text{Int}(Y) \Delta (Y - \text{Int}(Y))\).

It is enough to show that \(Y - \text{Int}(Y)\) is nowhere dense.

Given a basic open \((A,M)^{(\omega)}\), we have \(L \subseteq M\) with

\((A,L)^{(\omega)} \subseteq Y \cup Y^c\) (as \(Y\) is completely Ramsey).

- If \((A,L)^{(\omega)} \subseteq Y\) we have \((A,L)^{(\omega)} \subseteq \text{Int}(Y)\), so \((A,L)^{(\omega)}\) misses \(Y \setminus \text{Int}(Y)\) according to \(\text{Int}(Y)\) definition.

- If \((A,L)^{(\omega)} \subseteq Y^c\), certainly \((A,L)^{(\omega)}\) misses \(Y \setminus \text{Int}(Y)\) \(\blacksquare\)

**Remark**

Without Theorem 6, this proof would say

\(Y\) completely Ramsey \(\iff Y = \text{Open} \Delta \text{Nowhere dense}\).

But then we would not know that the Completely Ramsey set

form a \(\sigma\)-algebra.