Topics in Infinite Groups

1 Introduction

Example 1.1

i) X a set, $S(X)$ the permutation group of X

... ii) F a field, $GL_n(F)$ a group under multiplication

SUBGROUPS:

Proposition 1.2
$H \subseteq G$ is a subgroup of $G \Leftrightarrow H \neq \emptyset$ and $\forall a, b \in H$, $ab^{-1} \in H$. We write $H \leq G$ and $H < G$ if $H \neq G$.
proper subgroup

$I = \{e\}$, trivial subgroup

Proposition 1.3
(i) $L \leq H$ and $H \leq G \Rightarrow L \leq G$.
(ii) If $H_i \leq G$ $\forall i \in I$ then $\cap_{i \in I} H_i \leq G$.
arbitrary intersection

For $H_1, H_2 \leq G$, $H_1 \cup H_2$ is not generally a subgroup. But...

chain of subgroups

Proposition 1.4 (Ascending sequence of subgroups)
If $H_1 \leq H_2 \leq H_3 \leq \ldots \leq G$ (which means $H_n \leq G$ and $H_n \leq H_{n+1} \forall n$) then $\cup_{n=1}^{\infty} H_n \leq G$.

If we have groups G_n ($n \in \mathbb{N}$) for which $G_n \leq G_{n+1}$ and we form the set $X = \cup_{n=1}^{\infty} G_n$ then X is a group.

Direct limit

Example 1.5

i) S^1 under multiplication, $H_n = \{e^{2\pi i k/2^n} : 1 \leq k \leq 2^n\}$, $\bigcup_{n=1}^{\infty} H_n$ (quasicyclic, p-group, do for any p)

... ii) $G = S(\mathbb{Z})$, $H_n = \{$perms of $\{-n, \ldots, -n\}$, fix the rest$\} \cong S_{2n+1}$ $\leq H_{n+1}$

$\bigcup H_n = S_0(\mathbb{Z})$, finite support

Generators: Let $X \subseteq G$.

Definition 1.6
The subgroup $\langle X \rangle$ **generated by** X *is* $\cap H$ *over all* H *with* $X \subseteq H \leq G$. *It's the smallest subgroup of* G *containing* X. *We write* $\langle x_1, \ldots, x_k \rangle, \langle X, Y \rangle, \langle X_i : i \in I \rangle$ *etc.*

Definition 1.7
G *is* **finitely generated** *(fin.gen./f.g.) if* $G = \langle g_1, \ldots, g_k \rangle$. *Otherwise* G *is* **infinitely generated** *(inf.gen./i.g.).*

Example 1.8
$G = \langle g \rangle$ *means* G *is cyclic. Either* $G = \mathbb{Z}$ *or, if* G *has order* n, *we write* $G = C_n$.

If G has no non-trivial subgroups, $G = C_p$ or I.

Proposition 1.9
If $X \subseteq G$ *then the elements of* $\langle X \rangle$ *are (e and)*

$$x_1^{n_1} x_2^{n_2} \ldots x_k^{n_k} \text{ for } n_1, \ldots, n_k \in \mathbb{Z}$$

and $x_1, \ldots, x_k \in X$ *(but not necessarily distinct).*

So G finite $\Rightarrow$ G finitely generated $\Rightarrow$ G countable.
If we have an ascending sequence of subgroups $H_1 \leq H_2 \leq \ldots \leq G$ then either
$\exists N$ such that $H_N = H_{N+n} \forall n \in \mathbb{N}$ (terminates)
or (on throwing away duplicates) $H_1 < H_2 < \ldots$ (strictly ascending).

Definition 1.10
G *has* **max** *(satisfies the maximal condition) if every ascending sequence of subgroups terminates.*

like Noetherian

Theorem 1.11 *G has max $\Leftrightarrow$ H is finitely generated $\forall H \leq G$.*

Warning! We can have G finitely generated but $H < G$ with H infinitely generated. We will see examples later; indeed it can be argued that many finitely generated groups do not have max.

COSETS: If $H \leq G$ then the **left cosets** of H in G are the sets $gH = \{gh : h \in H\}$ for each $g \in G$.

Proposition 1.12 (Lagrange for infinite groups)
The left cosets of H in G form a partition of G and any left coset is in bijection with H.

Note: We have right cosets Hg in bijection with H too. Also there is a bijection from the set of left cosets to the set of right cosets given by gH goes to Hg^{-1} (not to Hg).

Definition 1.13
(i) The **index** *of H in G is the cardinality of the set of left (or right) cosets, written $[G : H]$ if finite.*
(ii) A **left (or right) transversal** *is a choice of left (or right) coset representatives, with exactly one for each coset.*

NORMAL SUBGROUPS: For $g, x \in G$ and $H \leq G$ the **conjugate** of x by g is $gxg^{-1} \in G$ and the **conjugate** of H by g is $gHg^{-1} \leq G$. Conjugacy is an equivalence relation.

Definition 1.14
The subgroup N is **normal** *in G (we write $N \trianglelefteq G$) if:*
(i) $gN = Ng \ \forall g \in G \Leftrightarrow$
(ii) $gNg^{-1} = N \ \forall g \in G \Leftrightarrow$
(iii) $gNg^{-1} \leq N \ \forall g \in G \Leftrightarrow$
(iv) N is a union of conjugacy classes of G.

Examples:
(i) $I \trianglelefteq G$ and $G \trianglelefteq G$.
(ii) If G is abelian then $H \trianglelefteq G$ for all subgroups H (but not the converse).
(iii) If $[G : H] = 2$ then $H \trianglelefteq G$.

Proposition 1.15 *(cf. Proposition 1.3)*
(i) If $N \trianglelefteq G$ and $H \leq G$ then $N \cap H \trianglelefteq H$.
But $M \trianglelefteq N$, $N \trianglelefteq G \not\Rightarrow M \trianglelefteq G$. not transitive, use D_8
(ii) $N_i \trianglelefteq G \ \forall i \in I \Rightarrow \cap_{i \in I} N_i \trianglelefteq G$.

Proposition 1.16 *(cf. Proposition 1.4)*
If $N_1 \leq N_2 \leq \ldots \leq G$ with $N_n \trianglelefteq G \ \forall n \in \mathbb{N}$ then $\cup_{n=1}^{\infty} N_n \trianglelefteq G$.

Let $X \subseteq G$.

larger ambient group matters

Definition 1.17 *(cf. Definition 1.6)*
The **normal closure** $\langle\langle X\rangle\rangle$, *or* $\langle\langle X\rangle\rangle_G$ *to be clear, of* X *in* G *is* $\cap N$ *over all* N *with* $X \subseteq N \trianglelefteq G$ *and is the smallest normal subgroup of* G *containing* X.

Note: We have $X \subseteq \langle X\rangle \leq \langle\langle X\rangle\rangle$ but $\langle\langle X\rangle\rangle$ can be much bigger than $\langle X\rangle$.

e.g. $X = H$, not normal in G. $\langle\langle X\rangle\rangle_H \neq \langle\langle X\rangle\rangle$

Proposition 1.18 *(cf. Proposition 1.9)*
If $X \subseteq G$ *then the elements of* $\langle\langle X\rangle\rangle$ *are (e and)*

$$g_1 x_1^{n_1} g_1^{-1} g_2 x_2^{n_2} g_2^{-1} \dots g_k x_k^{n_k} g_k^{-1}$$

for $n_1, \dots, n_k \in \mathbb{Z}$, $x_1, \dots, x_k \in X$ *and* $g_1, \dots, g_k \in G$ *(but not necessarily distinct).*

SET PRODUCTS: If $A, B \leq G$ then the **set product** $AB = \{ab : a \in A, b \in B\}$ is not in general a subgroup of G.

Proposition 1.19
(i) $AB \leq G \Leftrightarrow AB = BA$.
(ii) If so then $AB = \langle A, B\rangle$.
(iii) For $N \trianglelefteq G$ *and* $H \leq G$ *we have* $NH = HN$.

HOMOMORPHISMS: A function $\theta : G \to H$ for groups G, H is a **homomorphism** if $\theta(xy) = \theta(x)\theta(y)\ \forall x, y \in G$. It's an **isomorphism** if θ is bijective (both surjective=onto and injective=1-1) which is equivalent to $\exists$ an inverse $\theta^{-1} : H \to G$ (inverse here means two sided inverse), in which case θ^{-1} is unique and is also a homomorphism. We write $G \cong H$, they are then the same as abstract groups.

SETS AND FUNCTIONS: If X, Y are sets and $f : X \to Y$ is a function then for $U \subseteq X$ the **image** (or pushforward) of U is $f(U) = \{f(x) : x \in U\} \subseteq Y$, and for $V \subseteq Y$ the **inverse image** (or pullback) of V is $f^{-1}(V) = \{x \in X : f(x) \in V\} \subseteq X$.

Lemma 1.20
(i) $f^{-1}f(U) \supseteq U$ *and is equal if* f *is injective.*
(ii) $ff^{-1}(V) \subseteq V$ *and is equal if* f *is surjective.*

Theorem 1.21 *If* $\theta : G \to H$ *is a homomorphism and* $A \leq G$, $B \leq H$, *we have* $\theta(A) \leq H$, $\theta^{-1}(B) \leq G$ *and if* $B \trianglelefteq H$ *then* $\theta^{-1}(B) \trianglelefteq G$. *If* θ *is surjective then* $A \trianglelefteq G \Rightarrow \theta(A) \trianglelefteq H$.

Consequently the **image** $\theta(G)$ of θ is a subgroup of H and the **kernel** ker $\theta = \theta^{-1}(I)$ is a normal subgroup of G. In fact we have

Corollary 1.22 *For $\theta : G \to H$ with $A \leq G$, $B \leq H$ and $K = ker\,\theta$ we have $\theta^{-1}\theta(A) = AK$ and $\theta\theta^{-1}(B) = B \cap \theta(G)$.*

QUOTIENTS AND THE ISOMORPHISM THEOREMS: If $N \trianglelefteq G$ then the set of (left) cosets forms a group under the well defined multiplication $xN \cdot yN = (xy)N$. We say the group G/N is a **quotient** of G.

Theorem 1.23 (Homomorphism Theorem)
If $\theta : G \to H$ is a homomorphism then $G/ker\,\theta \cong \theta(G)$.

What are the subgroups of G/N? If $N \leq H \leq G$ then $N \trianglelefteq H$ and $H/N \leq G/N$.

Theorem 1.24 (Correspondence Theorem)
If $N \trianglelefteq G$ then the subgroups of G/N are exactly H/N for $N \leq H \leq G$ and the normal subgroups are exactly L/N for $N \leq L \trianglelefteq G$.

Note: If $\pi : G \to G/N$ is the natural homomorphism and $H \leq G$ then $\pi(H) = HN/N$.

Theorem 1.25 (Product Isomorphism Theorem)
If $H \leq G$ and $N \trianglelefteq G$ then $H/(H \cap N) \cong HN/N$.

Theorem 1.26 (Quotient Isomorphism Theorem)
Let $N, L \trianglelefteq G$ with $N \leq L$. Then $(G/N)/(L/N) \cong G/L$.

Definition 1.27 *Given $\theta : G \to H$ and $N \trianglelefteq G$ with $\pi : G \to G/N$ the natural homomorphism, we say that θ* **factors through** *G/N if $\exists\ \overline{\theta} : G/N \to H$ with $\theta = \overline{\theta}\pi$. We must have $N \leq ker\,\theta$ and this is sufficient by setting $\overline{\theta}(gN) = \theta(g)$.*

EXTENSIONS: If $G/N \cong Q$ we say that G is an **extension of** N **by** Q. We write G is N-by-Q but this does not necessarily determine G uniquely!

Lemma 1.28 *If G is A-by-(B-by-C) then $\Rightarrow$ G is (A-by-B)-by-C.*

Converse does NOT hold in general

<u>Proof</u> $G/A \cong Q$, $Q/B \cong C$

Thus B is N/A for $N \triangleleft G$ (correspondence)

So $C \cong \dfrac{G/A}{N/A} \cong G/N$ $\square$

Converse : $A_4 = (C_2 \text{ by } C_2) \text{ by } C_3$

Preserved Properties

	Finite	Countable	Cyclic	Abelian	F.g.	Max
Subgroups	✓	✓	✓	✓	✗	✓
Quotients	✓	✓	✓	✓	✓*	✓
Extensions	✓	✓	✗	✗	✓ i)	✓ ii)

GROUP PROPERTIES: These only depend on the abstract structure of the group. It is always worth asking whether a group property is preserved by (i) Subgroups, (ii) Quotients, (iii) Extensions (namely if $G/N \cong Q$ and N and Q both have this property then G does too).
For instance, what about finite, countable, cyclic, abelian, finitely generated, max?

$\theta(\langle g_1, \dots, g_n \rangle) = \langle \theta(g_1), \dots, \theta(g_n)$

Theorem 1.29 *The properties finitely generated and max are preserved by extensions.*

ACTIONS: We say the group G **acts** on the set X (on the left) if there is a function $\psi : G \times X \to X$ such that

$$\psi(e, x) = x \text{ and } \psi(g_1, \psi(g_2, x)) = \psi(g_1 g_2, x) \forall g_1, g_2 \in G, \forall x \in X.$$

Note for each $g \in G$ the function $x \mapsto \psi(g, x)$ is a permutation of X (put in g^{-1}, g and then g, g^{-1} to get an inverse).
Equivalently there is a homomorphism $\rho : G \to S(X)$ given by $\rho(g)(x) = \psi(g, x)$.
We say G acts **faithfully** (effectively) if ρ is injective. We can then unambiguously write $g(x)$ for $\rho(g)(x)$; we sometimes do this anyway. We say the action is (fixed point) **free** if $\rho(g)(x) = x \Rightarrow g = e$ (which implies faithful).
For $x \in X$ the **orbit** $\text{Orb}(x) = \{\rho(g)(x) : g \in G\} \subseteq X$ and the **stabiliser** $G_x = \{g \in G : \rho(g)(x) = x\} \leq G$. The orbits form a partition of X and the action is **transitive** if there's one orbit. If $y = \rho(g)(x)$ then $G_y = gG_xg^{-1}$ so stabilisers are rarely normal.

Theorem 1.30 (Orbit-Stabiliser)
If G acts on X then for $x \in X$ the sets $Orb(x)$ and $\{$left cosets of $G_x\}$ are in bijection.

Example 1.31
(i) G acts on itself via $\rho(g)(x) = gx$. This is transitive, and also free so $G \leq S(G)$.
(ii) G acts on itself by conjugation: $\rho(g)(x) = gxg^{-1}$. Here $Orb(x)$ is the **conjugacy class** *of x while the stabiliser is the* **centraliser** *$\{g \in G : gx = xg\}$ of x in G, written $C_G(x)$ or sometimes just $C(x)$ when clear. Note $\langle x \rangle \leq C(x)$ but it's not generally abelian.*

i) f.g groups preserved by extension

Proof

If $G/N = Q$, $N = \langle n_1, \dots, n_r \rangle$, $Q = \langle g_1 N, \dots, g_s N \rangle$, then $gN = g_0 N$ for $g_0 \in \langle g_1, \dots, g_s \rangle$ so $g = g_0 n$

ii) Max is preserved by extension

Proof Hany subgroup of G

$H/H \cap N \cong HN/N \leq G/N$ Use i)

f.g. max

We also have for $H \leq G$ the centraliser $C_G(H) = \{g \in G : gh = hg \ \forall h \in H\} = \cap_{h \in H} C_G(h)$. We set $C_G(G) = Z(G)$, the **centre** of G, which is abelian and normal.
Moreover G acts on the set of its subgroups by conjugation: $\rho(g)(H) = gHg^{-1} \leq G$. Then $\mathrm{Orb}(H)$ is the set of subgroups conjugate to H and the stabiliser is the **normaliser** $N(H) = \{g \in G : gHg^{-1} = H\} \leq G$. It is the largest subgroup of G in which H is normal. Also $C(H) \leq N(H)$.

AUTOMORPHISMS: An isomorphism (homomorphism) from G to G is an **automorphism** (**endomorphism**).

Example 1.32 *For any $g \in G$, $\alpha_g(x) = gxg^{-1}$ is an automorphism so $H \cong gHg^{-1}$. These are the* **inner** *automorphisms and form a group Inn(G) under composition.* Conjugacy

We have $\alpha_g = e \Leftrightarrow g \in Z(G)$ so $G/Z(G) \cong \mathrm{Inn}(G)$. Moreover all automorphisms form a group $\mathrm{Aut}(G)$, with $\mathrm{Inn}(G) \trianglelefteq \mathrm{Aut}\ (G)$ and the quotient is defined to be the **outer** automorphism group $\mathrm{Out}(G)$ of G.

Definition 1.33 *The subgroup C of G is* **characteristic** *in G if $\alpha(C) = C$ $\forall \alpha \in Aut(G)$ (but $\alpha(C) \leq C$ $\forall \alpha$ is enough), so $C \trianglelefteq G$.*
Like normality but stronger.

Proposition 1.34
(i) A characteristic in B, B characteristic in $C \Rightarrow A$ characteristic in C. transitive
(ii) A characteristic in B and $B \trianglelefteq C \Rightarrow A \trianglelefteq C$.
automorphism in C "descends" to automorphism in B.

DIRECT PRODUCTS: We can form the **direct product** $G_1 \times G_2$ from groups G_1, G_2 via $(x_1, x_2) \cdot (y_1, y_2) = (x_1y_1, x_2y_2)$. This is **external**, we can also do this **internally**.

Proposition 1.35 *If $M, N \trianglelefteq G$ with $MN = G$ and $M \cap N = I$ then $\theta : M \times N \to G$ given by $\theta(m, n) = mn$ is an isomorphism.*

We can extend this drastically:

Definition 1.36 *Given groups G_n for $n \in \mathbb{N}$, the* **Cartesian** *(unrestricted) product is the set of all sequences with nth component an element of G_n and pointwise multiplication; we write $\prod_{n \in \mathbb{N}} G_n$. The* **direct** *product $\times_{n \in \mathbb{N}} G_n$ is the subgroup of sequences which are eventually e.*

Note: If $G_n \neq I$ for infinitely many n then $\times G_n$ is infinitely generated and $\prod G_n$ is uncountable.

SEMIDIRECT PRODUCTS:

Definition 1.37 *Given groups G_1, G_2 and a homomorphism $\phi : G_2 \to Aut(G_1)$ then the* **semidirect product** *$G_1 \rtimes_\phi G_2$ is the set of ordered pairs with multiplication*

$$(x_1, x_2) \cdot (y_1, y_2) = (x_1(\phi(x_2)(y_1)), x_2y_2).$$

Example 1.38
(i) ϕ the trivial homomorphism gives the direct product.
(ii) Take $\mathbb{Z} = \langle z \rangle$ (written additively) and $C_2 = \{e, c\}$. Then $\mathbb{Z} \rtimes_\phi C_2$ with $\phi(c)(z) = -z$ is the **infinite dihedral group**.

Proposition 1.39 *(cf. Proposition 1.35)*
If $H \leq G$ and $N \trianglelefteq G$ with $NH = G$ and $N \cap H = I$ (so $G/N \cong H$ by 1.25) then $\theta : N \rtimes_\phi H \to G$ given by $\theta(n, h) = nh$ and $\phi(h)(n) = hnh^{-1} \in N$ is an isomorphism.

So again the internal and external versions are equivalent. There is another point of view:
If $G/N = Q$ with $\pi : G \to G/N$ the natural projection, we say that the extension **splits** if $\exists H \leq G$ such that $\pi : H \to Q$ is an isomorphism.
Now for $G = NH$ a semidirect product, π restricted to H is an isomorphism as $H \cap \ker \pi = I$. But a split extension implies $H \cap N = I$ and $NH = G$ as $\pi(H) = \pi(G)$, so they're the same.

Example 1.40
$SL(2, \mathbb{C}) = \{A \in GL(2, \mathbb{C}) : det\ A = 1\} \trianglelefteq GL(2, \mathbb{C})$ with quotient $\mathbb{C} - \{0\}$. If I_2 is the 2×2 identity matrix then $PSL(2, \mathbb{C}) = SL(2, \mathbb{C})/\{\pm I_2\}$. This extension does not split because we have elements g (of order four) in $SL(2, \mathbb{C})$ with $\pi(g)$ of order two in $PSL(2, \mathbb{C})$, but the only element of order 2 in $SL(2, \mathbb{C})$ is $-I_2$ with $\pi(-I_2) = e$.

order 2 $\Rightarrow$ det 1, $A = A^{-1}$

Example 1.41
Let $H = S_0(\mathbb{Z}) \leq S(\mathbb{Z})$
not f.g. But H is generated by $\{(n, n+1) : n \in \mathbb{Z}\}$ as any $h \in H$ is a perm of $-N, \ldots, N$ so h a product of transpositions.
Now consider $G = \langle H, f \rangle$, $f(z) = z+1$
Then $(n\ n+1) = f^n (0\ 1) f^{-n}$, so $G = \langle (0\ 1), f \rangle$
G has 2 generators, $H \leq G$, H not f.g.

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Topics in Infinite Groups ①

2 A Brief Guide to Abelian Groups

F.g. abelian groups.

<u>Theorem 2.1</u> (Structure theorem for f.g. abelian groups)

G f.g. abelian. Then:

(Rational) $G \cong \mathbb{Z}^r \times C_{d_1} \times \dots \times C_{d_s}$ with $1 < d_1 | d_2 | \dots | d_s$ (uniquely)

(Primary) $G \cong \mathbb{Z}^r \times P_1 \times \dots \times P_t$

where for each P_i we have a different prime p such that

$$P_i = C_{p^{e_1}} \times \dots \times C_{p^{e_j}} \text{ (uniquely)}, \quad 1 \leq e_1 \leq \dots \leq e_j$$

<u>Corollary 2.2</u>

G f.g. abelian $\Rightarrow$ G has max.

<u>Proof</u>

$\mathbb{Z}$ has max, so $\mathbb{Z}^k$ has max (extensions).

Now $\mathbb{Z}^{r+s} \twoheadrightarrow G$ in 2.1 so G has max. $\square$

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Topics in Infinite Groups (2)

Let G be a group. G is <u>torsion</u> if all elements have finite order. G is <u>torsion-free</u> if only e has finite order.

If G is f.g., let $d(G)$ = min. size of a generating set.

<u>Proposition 2·3</u>

For f.g. abelian $G \cong \mathbb{Z}^r \times C_{d_1} \times \ldots \times C_{d_s}$ as in 2·1, $d(G) = r+s$.

<u>Proof</u>

We have $d(G) \leq r+s$. Take some prime $p \mid d_1$, then $\exists \theta : G \twoheadrightarrow (C_p)^{r+s}$. $\theta(G)$ is a vector space over $\mathbb{F}_p$, and a generating set here is the same as a spanning set.

θ(gen. set for G) = gen. set for $\theta(G)$. $\square$

so $d(G) \geq r+s$

<u>Non f.g. Abelian Groups</u>

e.g. $\mathbb{Q}$, $\mathbb{Q}/\mathbb{Z}$, $\mathbb{R}$, $\mathbb{R}/\mathbb{Z}$

If p is a property preserved by subgroups, we say that G is <u>locally</u> p if H has p for all f.g. $H \leq G$.

<u>Examples</u>

1. $H = \langle \frac{p_1}{q_1}, \ldots, \frac{p_k}{q_k} \rangle \leq \mathbb{Q} \Rightarrow H \leq \langle \frac{1}{q_1 \cdots q_k} \rangle$

   $\therefore \mathbb{Q}$ is locally cyclic (torsion free)

2. $\mathbb{Q}/\mathbb{Z}$ is locally cyclic, and $p/q + \mathbb{Z}$ has finite order, so $\mathbb{Q}/\mathbb{Z}$ is locally finite (torsion).

Inv. I

If $A \times \mathbb{Z} \cong B \times \mathbb{Z}$ then we can have $A \ncong B$, but if this is abelian, then in fact $A \cong B$.

Proof (of 2nd part)

Suppose that $I' = A \times Y = B \times Z$, $Y, Z \cong \mathbb{Z}$.

i) If $A \subseteq B$, then $I'/A \cong \mathbb{Z} \twoheadrightarrow I'/B \cong \mathbb{Z}$

This induces an isomorphism. If $b \in B \setminus A$, b is in the kernel, but we know that the kernel must be trivial. Hence $A = B$.

ii) If $A \ncong B$, $B \ncong A$, then

$$\frac{A}{A \cap B} \cong \frac{AB}{B} \leq I'/B \cong \mathbb{Z}, \text{ and } B < AB.$$

So $A \cong (A \cap B) \times \mathbb{Z}$, $B \cong (A \cap B) \times \mathbb{Z}$ similarly.

Hence $A \cong B$.

Abelianisation

Let G be a group. Let $x, y \in G$.

The commutator $[x, y] := xyx^{-1}y^{-1} \in G$.

Definition 2.4

The <u>commutator</u> / <u>derived</u> subgroup G' of G is

$$\langle \{[x, y] : x, y \in G\} \rangle$$

Proposition 2.5

$G' \trianglelefteq G$ with G/G' abelian.

Proof

For α, an automorphism of G, $\alpha([x, y]) = [\alpha(x), \alpha(y)] \in G'$

So G' is <u>characteristic</u> in G. and characteristic $\Rightarrow$ normal

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Topics in Infinite Groups (2)

In G/G', $xyG' = yx[x^{-1}, y^{-1}]G'$ $\square$

<u>Corollary 2.6</u>

G/G' is the largest abelian quotient of G. If G/N is abelian then $G' \leq N$ (easy to see), so $G \twoheadrightarrow G/N$ factors through G/G'.

<u>Definition 2.7</u>

For any G, the <u>abelianisation</u> of G is this abelian quotient G/G'.

3 Free Groups

<u>Definition 3.1</u>

Let $X = \{a, b, \dots\}$ be a set of symbols. Let $X^{-1} = \{a^{-1}, b^{-1}, \dots\}$ be a set disjoint from X and in bijection with X.

We write $X^{\pm 1} = X \cup X^{-1}$. A <u>word</u> in $X^{\pm 1}$ is a finite sequence of elements of $X^{\pm 1}$ (letters) including ϕ.

Let W be the set of <u>all</u> words. Let $W_0 \subseteq W$ be the set of <u>reduced</u> words i.e. words containing no subword xx^{-1} or $x^{-1}x$.

We want to define the free group on X as the set of reduced words with multiplication $w_1 \cdot w_2 = w_1 w_2$, but proofs are difficult this way.

<u>Definition 3.2</u>

The free group $F(X)$ on set $X = \{x_i\}$ is a <u>subgroup</u> of $S(W_0)$ generated by elements :

symmetric group on reduced words

$$\mathcal{X}_i(w) = \begin{cases} x_i w & \text{if } w \text{ does } \underline{\text{not}} \text{ start with } x_i^{-1} \\ w' & \text{if } w = x_i^{-1} w' \end{cases}$$

$\mathcal{X}_i$ has image in W_0, and inverse

$$\mathcal{X}_i^{-1}(w) = \begin{cases} x_i^{-1} w & \text{if } w \text{ does } \underline{\text{not}} \text{ start with } x_i \\ w' & \text{if } w = x_i w' \end{cases}$$

<u>Proposition 3·3</u>

The map $M: W_0 \to F(X)$ given by replacing $x_i^{\pm 1}$ by $\mathcal{X}_i^{\pm 1}$ and multiplying in $F(X)$ is an injection.

<u>Proof</u>

If $M(w_1) = M(w_2)$ for $w_1, w_2 \in W_0$ then note that $M(w_i)(\phi) = w_i$ by (3·2) $\square$

<u>Corollary 3·4</u>

$M: W_0 \to F(X)$ is surjective, and given a word $w \in W$, if we delete all subwords xx^{-1}, $x^{-1}x$ in any order, we reach a unique word $w_0 \in W_0$.

<u>Proof</u>

i.e. in the obvious way

Extend to $M: W \to F(X)$, surjective by (1·9). Each deletion reduces the length of w, so we reach some $w_0 \in W_0$. Deletions do not change ~~the group element~~ $M(w)$, so $M(w) = M(w_0)$, so w_0 is unique by (3·3). $\square$

Word $\to$ Image in $F(X)$ under M

$\downarrow$

Unique reduced preimage

$\leadsto M: W_0 \to F(X)$ surjective

<u>Theorem 3·5</u>

(Universal Property of free groups) just any function

Any $f : X \to G$, a group, extends uniquely to a homomorphism $f^* : F(X) \to G$, so that $f^*(x_i) = f(x_i)$.

<u>Proof</u>

First, define $f^*(x_i^{-1}) = f(x_i)^{-1} \in G$

then let $f^*(t_1 \ldots t_k) = f(t_1) \ldots f(t_k)$ $\Big\}$ (*)

where $t_j \in \{x_i^{\pm 1}\}$ is represented by letter $L_j \in X^{\pm 1}$.

Then f^* is well-defined by (3·3), (3·4) and is a homomorphism by definition. Clearly, any such homomorphism must satisfy (*)

$\square$

$f : X \to Y$

$g : Y \to X$

$f \circ g = id_Y$

$g \circ f = id_X$

f injective :

$f(x_1) = f(x_2)$

$\Rightarrow g(f(x_1)) = g(f(x_2))$

$\Rightarrow x_1 = x_2$

f surjective :

$y \in Y$

$g(y) \in X$

$f(g(y)) = y$

$\therefore f$ bijective (similarly for g)

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Proposition 3.6

If $F(X)$ and $F(Y)$ are free groups on X, Y, then

$F(X) \cong F(Y) \iff |X| = |Y|$

Proof

$|X| = |Y|$ so there is a bijection

($\Leftarrow$) If $f: X \to Y$ is a bijection then we can extend to a homomorphism in the obvious way

$f^*: F(X) \to F(Y)$ and $(f^{-1})^*: F(Y) \to F(X)$.

But $(f^{-1})^* f^*$ fixes X so this is the identity homomorphism and the same for $f^*(f^{-1})^*$ so bijective.

($\Rightarrow$) For any group G, let $S_G \triangleleft G$ be $\langle g^2 : g \in G \rangle$.

Then G/S_G is abelian with all non-identity elements of order 2, thus is a vector space over $\mathbb{F}_2$. Now the image of X in $F(X)/S_{F(X)}$ is linearly independent and spans so $|X|$ is the dimension. $\square$.

obvious

uniqueness of dimension $\Rightarrow |X| = |Y|$

$ab S_G = ba S_G$
$\Updownarrow$
$a^{-1}b^{-1}ab \in S_G$
$\parallel$
$i^2(ab^{-1})^2 b^2$

Lin. dependent in G/S_G

$\Rightarrow$ Lin. Dependent in G

We can define F_n (the free group of rank n) when $|X| = n$.

$F_0 = I$, $F_1 = \mathbb{Z}$, but if $a, b \in X$, $a \neq b$, then $ab \neq ba$ in $F(X)$ so these are non-abelian.

In fact $\langle a, b \rangle = F_2 \leq F(X)$.

Corollary 3.7

Every (finitely generated) group is a quotient of a (finitely generated) free group.

Proof

If $G = \langle g_i : i \in I \rangle$ then take $X = \{x_i : i \in I\}$ and use 3.5.

$f: X \to G$, $x_i \to g_i$ extends to a homomorphism f^* $\square$

$G \cong \dfrac{F(X)}{\ker f^*}$

Corollary 3.8

If $G \xrightarrow{\theta} F_n$ then $F_n \leq G$ and $G = \ker\theta \rtimes F_n$

Proof

possible since θ is a surjection

Take $g_1, \dots, g_n \in G$ with $\theta(g_i) = x_i$ then extend to a map $f : F(X) \to G$

$f(x_i) = g_i$. We have θf is the identity homomorphism (unique)

so $f : F_n \to G$ is an isomorphism from F_n to $f(F_n)$ since θ acts as an inverse

not necessarily all of G

For $K = \ker\theta$, $Kf(F_n) = G$ and $K \cap f(F_n) = I$ (1.39) $\square$

Definition 3.9

We say that the set $S = \{s_i\} \subseteq F(X)$ is a free basis for $F(X)$ if $\langle S \rangle = F(X)$ (generates) ("spans") c.f. basis of a vector space

and for any reduced word $w_0 \neq \emptyset$ on $S^{\pm 1}$

we have $w_0 \neq id$ when evaluated in $F(X)$. (free) ("linearly independent")

Proposition 3.10

Free bases for $F(X)$ have cardinality $|X|$.

Proof

Given a free basis S for $F(X)$, define a homomorphism $f^* : F(S) \to F(X)$ by extending $s \in S \mapsto s \in F(X)$.

Then f^* is surjective (generates) and injective (free).

Then by 3.6, $F(X) \cong F(S) \Rightarrow |S| = |X|$ $\square$

Proposition 3.11

The automorphisms of $F(X)$ are exactly (extensions of) bijective functions $f : X \to S$, S a free basis.

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Proof

~~considering $w \in F(X)$ as a function?~~ w is a word in the x_i?

An automorphism α must send a free basis bijectively to a free basis as $\alpha(w(x_i)) = w(\alpha(x_i))$ and it is determined by this. Moreover, given f, extend to $f^* : F(X) \to F(X)$ uniquely.

so all automorphisms send free basis to free basis

Then $f(X) = S$ means that f^* is surjective and S free says that f^* is 1-1. so every free basis mapping gives an automorphism $\square$

Definition 3.12

i.e. w cannot be "shortened" by conjugation

A word in $X^{\pm 1}$ is <u>cyclically reduced</u> if it is reduced and the first and last letters of w are not inverses.

We can write any reduced word $w_0 = u \,|\, w_1 \,|\, u^{-1}$ where w_1 is cyclically reduced and $|$ means no cancellation between 2 words.

Proposition 3.13

If w, w' are cyclically reduced then they are conjugate $\iff$ w' is a cyclic permutation of $w = l_1 | l_2 | \ldots | l_n$ for $l_i \in X^{\pm 1}$

Proof

If $w' = \overset{(i)}{c}w\overset{(ii)}{c^{-1}}$ (c reduced, $\neq \emptyset$) then as w' is cyclically reduced, we have some cancellation at (i), or (ii), but not both (as w is cyclically reduced). WLOG, say $c = d \,|\, l_1^{-1}$ necessary

$w = l_1 | v$, then $w' = dv l_1 d^{-1}$ but vl_1 is a cyclic permutation of w so either $d = \emptyset$ or we continue the process ~~(N.B. Contradiction~~ (N.B. terminates as length (c) decreases each time) $\square$

converse: $w = l_1 | \ldots | l_n$

$w' = l_k | \ldots | l_n | l_1 | \ldots | l_{k-1}$

$w' = (l_{k-1}^{-1} \ldots l_1^{-1})\, w\, (l_1 \ldots l_{k-1})$

Example 3·15

Take F_2 free on a, b. Let $H_n = \langle a, bab^{-1}, \dots, b^n a b^{-n} \rangle \leq F_2$

Corollary 3·14

A free group is torsion-free.

Proof

Any reduced word w_0 $(\neq \phi)$ $= u|w|u^{-1}$ where w is cyclically reduced. But for $n \geq 0$, $w_0^n = u|w^n|u^{-1}$ so we have no cancellation, so $w_0^n \neq \phi$.

Example 3·15 (continued)

By 1·9, if $H_n \ni h$, then $h = b^{i_1} a^{k_1} b^{i_2 - i_1} a^{k_2} b^{i_3 - i_2} \dots b^{-i_m}$ for $i \leq i_j \leq n$. Hence $b^{n+1} a b^{-(n+1)} \notin H_n$.

So $H = \bigcup H_n$ is not finitely generated by 1·11. because H does not have max.

Corollary 3·16

A finitely generated group G containing a non-abelian free group ($\Leftrightarrow F_2 \leq G$). ~~So G~~ does not have max. using H_n $\square$

Free Products

Definition 3·17

Let $\{G_\lambda : \lambda \in \Lambda\}$ be an indexed family of groups.

A <u>reduced sequence</u> in $\{G_\lambda\}$ is a finite sequence $g_1, \dots, g_n$ where $g_j \in \coprod_{\lambda \in \Lambda} G_\lambda$. Each $g_j \neq e$ and no successive g_j, g_{j+1} are in the same G_λ.

Let $\mathcal{R}$ be the set of <u>all</u> reduced sequences (including ϕ).

A the set of all sequences in $\coprod_{\lambda \in \Lambda} G_\lambda$

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Definition 3.18

The free product $\mathop{*}_{\lambda\in\Lambda} G_\lambda$ is the subgroup of $S(R)$ generated by elements $r_{(g,\lambda)}$ for $\lambda\in\Lambda$ and $(g,\lambda)\in G_\lambda\setminus I$, where

$$r(g,\lambda)(g_1\dots g_n) = \begin{cases} (g,\lambda)\, g_1\dots g_n & \text{if } g_1\notin G_\lambda \\ \underbrace{((g,\lambda)g_1)\, g_2\dots g_n}_{\text{in } G_\lambda\text{, remove if } e.} & \text{if } g_1\in G_\lambda \end{cases}$$

with $r^{-1}(g,\lambda) = r(g^{-1},\lambda)$

Proposition 3.19

The function $f: R \to *G_\lambda$ given by

$f(g_1,\dots,g_n) = rg_1\circ\dots\circ rg_n$ is a bijection.

Proof

If $g_1, g_2\in G_\lambda$ then $rg_1\circ rg_2 = rg_1g_2$ so gather rs from some group to get a surjection (remove e).

Then $f(g_1\dots g_n)(\emptyset) = rg_1\dots rg_n(\emptyset) = g_1\dots g_n$ $\square$

Proposition 3.19

The function $f: A \to *G_\lambda$ strictly this is $r(g,\lambda)$

$f(g_1\dots g_n) = rg_1\circ\dots\circ rg_n$ is a bijection on R

Proof

If $g_1, g_2\in G_\lambda$, the same group, then $rg_1\circ rg_2 = rg_1g_2$

Collect a large enough set of r's so that the map is surjective.

$f(g_1\dots g_n)(\emptyset) = rg_1\dots rg_n(\emptyset) = g_1\dots g_n$

so this map is injective.

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Edits

3.17) $\mathcal{A}$ the set of all sequences in disjoint union.

3.19) $f : \mathcal{A} \to * G_\lambda$ given $f(\ldots) = r$ $(r_{(e,\lambda)} = Id_{S(R)})$ restricts to a bijection from $\mathcal{R}$.

Note

$\iota_\lambda : G_\lambda \to \ast_{\lambda \in \Lambda} G_\lambda$ given by $\iota_\lambda(g) = (g, \lambda)$ is an injective homomorphism.

Theorem 3.20 (Universal Property)

Compare with 3.15

For any group H and any collection of homomorphisms $\theta_\lambda : G_\lambda \to H$, $\exists!\ \theta : \ast_{\lambda \in \Lambda} G_\lambda \to H$, such that $\theta_\lambda = \cancel{\theta_{\lambda}}\ \forall \lambda$ $\quad \theta_\lambda = \theta \circ \iota_\lambda$

Proof

Define $\theta(g_1 \ldots g_n) = \theta_{\lambda_1}(g_1) \ldots \theta_{\lambda_n}(g_n)$ where $g_i \in G_{\lambda_i}$.

Similarly, this is a homomorphism and is unique. $\square$

If $F(X)$ is free on $X = \{x_i : i \in I\}$ then it is $\ast_{i \in I} G_i$ where $G_i = \{x_i^n : n \in \mathbb{Z}\}$ infinite cyclic.

Note that $G_1 * G_2$ is infinite if $G_1, G_2 \neq I$, as $(g_1 g_2)^n \neq e$ for $n > 0$ and $g_2 g_1 \neq g_1 g_2$.

Suppose that X is a topological space and G the group of homeomorphisms of X. We say that $S \subseteq X$ is a <u>G-packing</u> if $g(S) \cap S = \emptyset$ $\forall g \in G \setminus I$.

Theorem 3.21 (Klein, 1883)

If $G_1, G_2 \leq \mathrm{Homeo}(X) = G$ with G_i packings S_i such that $S_1 \cup S_2 = X$ and $S_1 \cap S_2 \neq \emptyset$ then $G = \langle G_1, G_2 \rangle = G_1 * G_2$.

Proof

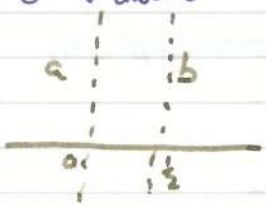
Note for $x \in S_1$, $g(x) \notin S_1$ for any $g \in G_1 \setminus I$ so $g(x) \in S_2$.

Take $s \in S_1 \cap S_2$ and a reduced sequence $g_1, \dots, g_n$ with (WLOG) $g_n \in G_1$. Then $g_n(s) \in S_2 \setminus S_1$, $g_{n-1} g_n(s) \in S_1 \setminus S_2$, and so on, so that $g_1 \dots g_n(s) \neq s$. Thus, the homomorphism in

(3·20) $\theta : G_1 * G_2 \twoheadrightarrow \langle G_1, G_2 \rangle$ is an isomorphism. $\square$

the above ensures injectivity as θ has trivial kernel

Examples



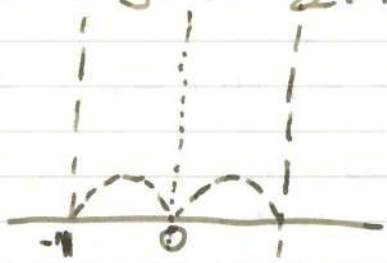
1. Let $X = \mathbb{R}^2$ with reflections a, b in lines

Then $\langle a, b \rangle = \langle a, ba \rangle = C_2 * C_2$

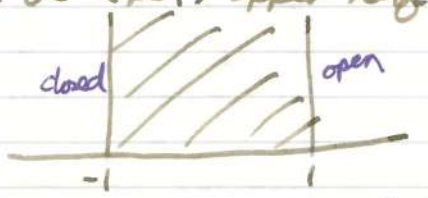
This is an infinite dihedral group (1·38) as $ba(x) = x + 1$ and $a(ba)a^{-1} = (ba)^{-1}$

2. Möbius transformations, $f(z) = \frac{az+b}{cz+d}$, $a, b, c, d \in \mathbb{C}$, $ad - bc \neq 0$ is a bijection of $\mathbb{C} \cup \{\infty\}$ with inverse $f^{-1}(z) = \frac{dz - b}{-cz + a}$, so they form a group.

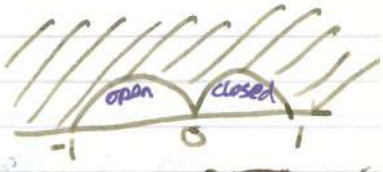
$f(z) = z + 2$, $g(z) = \frac{z}{2z+1}$ both preserve the (open) upper half plane U.



$S_1 =$



$S_2 =$



Proposition 3·23

$F = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$, $G = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \in SL_2(\mathbb{Z})$, $SL_2(\mathbb{Z}) \cong \langle F, G \rangle$

Proof

$f, g \in PSL_2(\mathbb{Z}) \xleftarrow{\pi} SL_2(\mathbb{Z}) / \{\pm I\}$, and $\langle f \rangle, \langle g \rangle$ are infinite, so $\langle f, g \rangle = F_2$ by 3·21.

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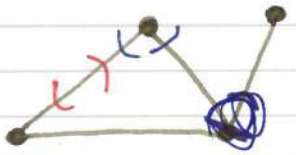
Now send $f, g \mapsto F, G$ and extend (3·5) to Θ with

$\pi\Theta = id$, so $\Theta(F_2) = F_2$ $\square$

<u>Definition 3·24</u>

A graph Γ (a 1-d CW complex) is made up of a set V of vertices with discrete topology and a set E of edges I_α (with Interior $(I_\alpha) = E_\alpha$). Each I_α is a copy of $[0,1]$. The edge endpoints are attached to points in V, giving $f_0, f_1 : E \to V$ so that $\Gamma = V \coprod_\alpha E_\alpha \Big/ \begin{matrix} I_\alpha(0) = f_0(I_\alpha) \\ I_\alpha(1) = f_1(I_\alpha) \end{matrix}$

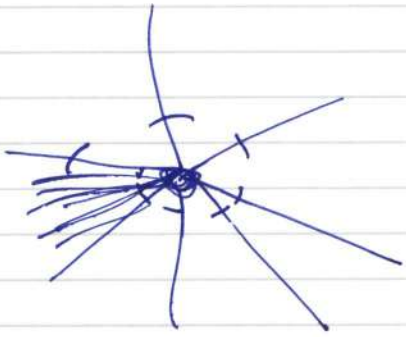


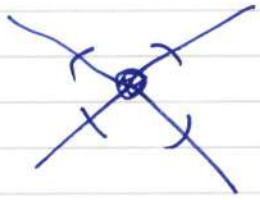
1. $S \subset \Gamma$ is open (closed) $\Leftrightarrow$ $S \cap \bar{E}_\alpha$ open (closed) $\forall \alpha$
2. A basic open ~~set~~ neighbourhood of $v \in \Gamma$ is 

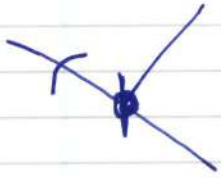
   Γ is locally path connected, locally contractible.

   (connected $\Leftrightarrow$ path connected)
3. Basic open sets for Γ are these + open intervals in E_α.

not allowed this type!







Topics in Infinite Groups
Topological Background

We follow the books Lee and Hatcher as in the course summary.

X is a topological space which is always assumed to be path connected and **locally path connected**:
$\forall x \in X$ and $\forall$ open $U \subseteq X$ with $x \in U$, we have an open path connected set P with $x \in P \subseteq U$.

Let $f, g : X \to Y$ be continuous maps (and let $A \subseteq X$).
A **homotopy** between f and g (relative to A, written rel A) is a continuous map $H : X \times [0,1] \to Y$ with $H(.,0) = f$ and $H(.,1) = g$ (with $H(a,t) = f(a)$ $\forall a \in A$ and $\forall t \in [0,1]$, so f and g must agree on A). This is an equivalence relation, written $f \simeq g$ (or $f \simeq g$ rel A).
A (strong) **deformation retraction** of X onto $A \subseteq X$ is a homotopy rel A from Id_X to $r : X \mapsto X$ with $r(X) \subseteq A$ and $r|_A = Id_A$.

For $x_0 \in X$ the **fundamental group** $\pi_1(X, x_0)$ is the group of homotopy classes of closed paths γ with start and end x_0 (in other words $\gamma : [0,1] \to X$ is continuous with $\gamma(0) = \gamma(1) = x_0$). Changing the basepoint x_0 "doesn't matter" as we obtain an isomorphic group, written $\pi_1(X)$.

Any continuous map $f : X \to Y$ induces a homomorphism $f_* : \pi_1(X) \to \pi_1(Y)$. If X is homeomorphic to Y (which is also written $X \cong Y$) then $\pi_1(X) \cong \pi_1(Y)$.

X is **homotopy equivalent** to Y (written $X \simeq Y$) if there exist continuous maps $f : X \to Y$ and $g : Y \to X$ such that $gf \simeq \mathrm{Id}_X$ and $fg \simeq \mathrm{Id}_Y$. If so then $f_* : \pi_1(X) \to \pi_1(Y)$ is an isomorphism.

X is **contractible** if $X \simeq \{x\}$, which implies that X is **simply connected** (meaning that $\pi_1(X)$ is the trivial group).

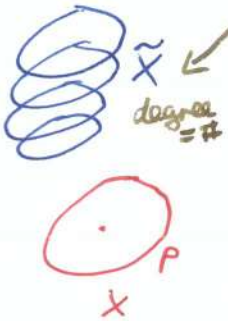
Seifert - Van Kampen Theorem (Hatcher Theorem 1.20 in the case where the intersections are simply connected): If $X = \cup_\alpha A_\alpha$ where all of the A_α are path connected and open in X, with $x_0 \in \cap_\alpha A_\alpha$ and each pairwise

Can take free product of fundamental groups of simple regions to get whole region.

and triple intersection $A_{\alpha_1} \cap A_{\alpha_2}$; $A_{\alpha_1} \cap A_{\alpha_2} \cap A_{\alpha_3}$ is simply connected then

$$\pi_1(X) \cong *_\alpha \pi_1(A_\alpha).$$

If $A \subseteq X$ and $\iota : A \to X$ is the inclusion map then sadly $\iota_* : \pi_1(A) \to \pi_1(X)$ might not be injective or surjective (see picture). But in a deformation retraction $X \simeq A$ (by taking $f = r$ and g as inclusion of $A = Y$ in X) so X and A have isomorphic fundamental groups.



A continuous map $p : \tilde{X} \to X$ is a **covering map** (with $\tilde{X}$ a **covering space** for X) if $\forall x \in X$ there exists an open neighbourhood V of x with $p^{-1}(V)$ a disjoint union of open sets in $\tilde{X}$, each of which is mapped homeomorphically by p onto V. As X is connected, the cardinality of $p^{-1}(x)$ (which is non-zero as we assume $\tilde{X}, X \neq \emptyset$) is constant: the **degree** or **number of sheets**.

Given a path $\gamma : [0,1] \to X$ and a point $\tilde{x}$ above $\gamma(0)$, there exists a unique path $\tilde{\gamma} : [0,1] \to \tilde{X}$ lifting γ (namely $p\tilde{\gamma} = \gamma$). Moreover $p_* : \pi_1(\tilde{X}) \to \pi_1(X)$ is injective. Monodromy

Now assume X is **locally contractible**.

Classification of Coverings Theorem: For each $H \leq \pi_1(X)$ there exists a cover $p : \tilde{X} \to X$ with $p_*(\pi_1(\tilde{X})) = H$ (this cover is "unique"). The degree is the index of H in $\pi_1(X)$.

and some covering space $\tilde{X}$

A **deck transformation** of the cover $p : \tilde{X} \to X$ is a homeomorphism ϕ of $\tilde{X}$ such that $p\phi = p$. They form a group D and for any $x_0 \in X$ the action of D on $p^{-1}(\{x_0\}) \subseteq \tilde{X}$ is transitive if and only if $p_*\pi_1(\tilde{X}, \tilde{x}_0)$ is normal in $\pi_1(X, x_0)$, in which case we say p is a **regular** cover.

c.f. Galois Theory!

03/01/14 Topics in Infinite Groups (5)

2pm, M 17 Feb

A <u>subgraph</u> of Γ is a union Δ of edges and vertices such that $e_\alpha \in \Delta \Rightarrow \overline{e_\alpha} \in \Delta$.

an edge

<u>Lemma 3.25</u>

If $\tilde{X}$ covers the (connected) graph Γ then $\tilde{X}$ is a graph with vertices and edges the lifts of those in Γ.

connected!!

<u>Proof</u>

Let $V(\tilde{X}) = p^{-1}(V)$ for $p : \tilde{X} \to \Gamma$, and for edges take the map from I_α into Γ and $\tilde{v}$ above $f_0(I_\alpha)$ to get a unique lift. Thus $\tilde{X}$ is a graph and the topologies agree on basic open sets, so they are the same. $\square$

If S is a simple graph (no self loops or multi-edges), an <u>edge</u> path in S is a finite sequence of vertices such that any 2 consecutive vertices span an edge in S. A <u>cycle</u> $v_0, \dots, v_n$ is a closed edge path ($v_0 = v_n$). A <u>tree</u> is a connected simple graph with no reduced cycles.

<u>Proposition 3.26</u>

Given any connected graph Γ, any vertex v_0 in Γ, there exists a subgraph $\Delta \simeq \{v_0\}$ such that Δ contains all of $V(\Gamma)$.

i.e. Γ has a spanning tree which contracts to v_0

<u>Proof</u>

Let $\Gamma_0 = \{v_0\} \subseteq \Gamma_1 \subseteq \Gamma_2$ be a sequence of subgraphs where Γ_{i+1} is $\Gamma_i \cup$ edges $\overline{e_\alpha}$ for all $e_\alpha \subseteq \Gamma \setminus \Gamma_i$ with an endpoint in Γ_i. i.e. $\Gamma_{i+1} = \Gamma_i + (\text{neighbours of } \Gamma_i)$

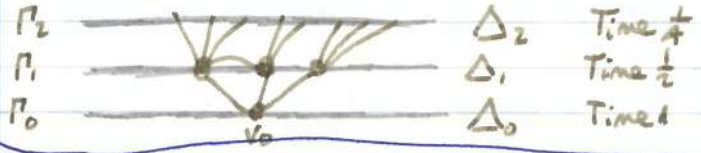
Then $\bigcup \Gamma_i$ is open (neighbourhood of a point in Γ_i is in Γ_{i+1})

and closed in Γ (union of closed edges), so is all of Γ.

we used AC

Next $\Delta_0 = \Gamma_0 \subseteq \Delta_1 \subseteq \Delta_2 \subseteq \dots$ where Δ_i has the same ~~edges~~ vertices as Γ_i but $\Delta_{i+1} = \Delta_i$ union <u>one</u> edge from each $v \in \Gamma_{i+1} \setminus \Gamma_i$ to Δ_i.

Now Δ_{i+1} deformation retracts onto Δ_i. So set $\Delta = \bigcup \Delta_i$ (contains $V(\Gamma)$). Δ deformation retracts to Δ_0 by performing homotopies in the interval $\left[\frac{1}{2^{i+1}}, \frac{1}{2^i}\right]$, with homotopy $\Delta \to \Delta$ continuous $\square$



No reduced cycle in Δ.

<u>Corollary 3.27</u>

A tree T is contractible.

<u>Proof</u>

Apply 3.26 to get $\Delta \subseteq T$ contractible. If edge e is spanned by $v \neq w$, e not in Δ then take reduced edge paths $v_0, \dots, uv$ and $wx \dots v_0$ in Δ. Note that $u \neq w$, $v \neq x$ as $e \notin \Delta$. So $v_0 \dots vw \dots v_0$ is a reduced cycle in T ※ $\square$

So $\Delta = T$, Δ contractible

<u>Theorem 3.28</u>

The fundamental group of a (connected) graph Γ is free.

<u>Proof</u>

Let $T \subseteq \Gamma$ (WLOG simple) be Δ (in 3.26) so this is a tree, with $\{e_\alpha : \alpha \in A\}$ the edges in $\Gamma \setminus T$ and w_α, w_α' endpoints of e_α. For each α, take a reduced cycle

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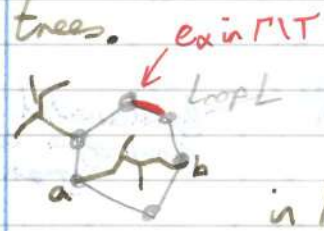
C_α from v_0, including $w_\alpha w_\alpha'$ so that $C_\alpha = \overbrace{v_0 u_1 u_2 \dots w_\alpha}^{\in T} \overbrace{w_\alpha' \dots v_1 v_0}^{\in T}$

First assume one edge. If C has $u_i = v_j$ in it (in $\Gamma \setminus T$?) then replace with $u_i \dots v_j$ which still contains $w_\alpha w_\alpha'$, otherwise it is a reduced cycle in T. So now we have (renamed) $v_0 \dots v_0$ with no repeats giving a loop $L \cong S^1$, so $\pi_1 L \cong \mathbb{Z}$.

But $\Gamma \cong L$ as the components K of $\overline{\Gamma \setminus L}$ are in T and so are trees.

~~is a second loop satisfying the same conditions as L?~~



If $a \neq b$, $a, b \in K \cap L$, then go from a to b in K then go from a to b in K the back via L (no via w_α, w_α') to get reduced cycles in T.

Thus by 3·27 we can deformation retract each K (simultaneously) onto the point in $K \cap L$, giving $\pi_1(\Gamma) = \mathbb{Z}$.

For general Γ, let m_α be a "midpoint" of e_α and set

$A_\alpha = (\Gamma \setminus \bigcup_{a \in \alpha} m_a) \cup m_\alpha$ i.e each A_α is Γ disconnected at all edges $\Gamma \setminus \Delta$ except 1.

Then A_α is open in Γ and path connected with

$A_\alpha \cap A_\beta (\cap A_\gamma) = \Gamma \setminus \bigcup_a m_a$ which deformation retracts onto T so that it is simply connected by 3·27). Also, A_α retracts onto $T \cup e_\alpha$. So $\pi_1(A_\alpha) \cong \mathbb{Z}$ giving (Seifert-van Kampen)

$\pi_1(\Gamma, v) \cong \mathop{*}_\alpha \mathbb{Z}$ $\square$

<u>Corollary 3·29</u>

For every free group $F(X)$ $\exists$ a simple graph Γ with $\pi_1 \Gamma = F(X)$

Proof "bouquet of circles"

Take a loop L_α for each $x_\alpha \in X$, all join at v, turn it into a triangle and use 3·28 $\square$

Theorem 3·30 (Nielsen/Schreier)

A subgroup of a free group is free.

Proof

For $H \leq F(X)$, use 3·29), subgroups $\Leftrightarrow$ coverings ($\tilde{X} \to X$, $\pi_1(\tilde{X}) = H$), 3·25) and 3·28). H is free. $\square$

lifting gives a connected graph — fundamental group of a connected graph is free.

Theorem 3·31 (Nielsen-Schreier index formula)

If H has index i in the free group F_n then H has rank $i(n-1)+1$.

Proof

For a finite graph Γ, define $\chi(\Gamma) = |V| - |E|$

separately $|\tilde{V}| = i|V|$, $|\tilde{E}| = i|E|$

If $p : \tilde{\Gamma} \to \Gamma$ has degree i then $\chi(\tilde{\Gamma}) = i\chi(\Gamma)$ by 3·25.

Take Γ in 3·29) with $\pi_1\Gamma = F_n$ and $\chi(\Gamma) = 1-n$.

For a subgroup H of index i in F_n with $\pi_1(\tilde{\Gamma}) = H$, we have $\chi(\tilde{\Gamma}) = i(1-n)$. Now in 3·28 note $\chi(\Delta) = 1$, so the set of edges $\{e_\alpha\}$ for $\tilde{\Gamma} \setminus \Delta$ numbers $i(n-1)+1$ so this is the rank of H. $\square$

for counting $\left(-\chi(\tilde{\Gamma}) + \chi(\Delta)\right)$

$\tilde{\Gamma}/\Delta$ is what is actually happening

$\tilde{E}$ − Edges of Δ = edges contributing to fundamental group

Δ spanning tree

Notes

i) $F_2 \leq F_3$, <u>and</u> $F_3 \leq F_2$.

ii) If $N \triangleleft F(X)$, it can be shown that if N has infinite index and $N \neq I$, then N has infinite rank.

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Topics in Infinite Groups (6)

4 Presentations of Groups

By 3.7 we can write any group G as $F(X)/N$ where (the image of) X is a generating set for G.

Definition 4.1

A <u>presentation</u> $\langle X \mid R \rangle$ for a group G is a set X and a subset R of $F(X)$ such that $G \cong \frac{F(X)}{\langle\langle R \rangle\rangle}$

Elements of X are <u>generators</u>, of R are <u>relators</u>.

Theorem 4.2 (von Dyck)

If $G = \langle X \mid R \rangle$ then Q is a quotient of G

$\iff Q \cong \langle X \mid R \cup S \rangle$ (quotient = more relators)

Proof

($\Leftarrow$) Have $F(X) \overset{\pi}{\twoheadrightarrow} \langle X \mid R \cup S \rangle$ factors through G as

$\langle\langle R \rangle\rangle \leq \langle\langle R \cup S \rangle\rangle$ trivial

($\Rightarrow$) Suppose $Q = G/L$ for $G = F(X)/M$ where $M = \langle\langle R \rangle\rangle$

Then $L \underset{\text{subgroup correspondence theorem}}{=} NM/M$ for $N \trianglelefteq F(X)$ so $Q \cong F(X)/NM$

Take any S with $\langle\langle S \rangle\rangle = N$.

Then $\langle\langle R \cup S \rangle\rangle = \langle\langle N \cup M \rangle\rangle = MN$.

So $Q \cong \langle X \mid R \cup S \rangle$ $\square$

Definition 4.3

G is <u>finitely presented</u> (f.p) if

$G = \langle x_1, \dots, x_n \mid r_1, \dots, r_m \rangle$

Proposition 4.4 (B. Neumann 1937)

If $G = \langle x_1, \dots, x_n \mid r_1, \dots, r_m \rangle = \langle y_1, \dots, y_k \mid s_1, \dots \rangle$.

Then $G = \langle y_1, \dots, y_k \mid s_1, \dots, s_L \rangle$ for some L.

<u>Proof</u>

We set $y_i = v_i(x_1, \dots, x_n)$, $x_i = w_i(y_1, \dots, y_k)$,

Write as words in the other set of generators

thinking of x_i, y_i as elements of G.

Thus $y_i = v_i(w_1(y_j), \dots, w_n(y_j))$ (j runs over $1 \le j \le k$)

and $r_i(w_1(y_j), \dots, w_n(y_j)) = e$ in G.

Let $N = \langle\langle s_1, s_2, \dots \rangle\rangle \triangleleft F(\{y_i\})$ and form

$\bar{y}_i$ behave as if generated by x_i

What we think G "should be" in terms of $\{y_i\}$

$\bar{G} = \langle \bar{y}_1, \dots, \bar{y}_k \mid \bar{y}_i = v_i(w_1(\bar{y}_j), \dots, w_n(\bar{y}_j)), r_i(\dots) \rangle$

$\bar{y}_i$ satisfy same relations

so $\exists\ \theta : \bar{G} \to F(\{y_i\})/N$, $\bar{y}_i \mapsto y_i$

as each relator holds in G, since $F(\{\bar{y}_i\}) \twoheadrightarrow F(\{y_i\})/N \ (\cong G)$

factors through $\bar{G}$.

Now take $\varphi : \langle x_1, \dots, x_n \mid r_1, \dots, r_m \rangle \twoheadrightarrow \bar{G}$ $\quad G \cong$

by $\varphi(x_i) = \bar{x}_i = w_i(\bar{y}_1, \dots, \bar{y}_k)$ $\quad G \xrightarrow{\varphi} \bar{G} \xrightarrow{\theta} G$

Then $\theta\varphi(x_i) = x_i$, φ onto, so θ is an isomorphism.

Therefore $N = \langle\langle \text{finite set} \rangle\rangle$ because all normal subgroups of $H \triangleleft F(X)$ N are finitely normally generated

But $N = \bigcup_{i=1}^{\infty} \langle\langle s_1, \dots, s_i \rangle\rangle$ (use max property) $\square$

<u>Proposition 4.5</u>

$G = \langle a, b \mid \overbrace{[a^{2n+1} b a^{-(2n+1)}, b]}^{c_n}, n \in \mathbb{N} \rangle$ <u>is not f.p.</u>

check

<u>Proof</u>

In A_j (alternating group), let $\alpha = (1\,2 \dots j)$ and $\beta = (1\,2\,3)$.

Then $\alpha^k \beta \alpha^{-k}$ commutes with β if $3 \le k \le j-3$, but not if $k = j-2$. Thus in A_{2l+3} we have $c_1, \dots, c_{l-1} = e$ but $c_l \ne e$.

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Topics in Infinite Groups (6)

G has an ascending chain of normally generated subgroups

So $c_i \notin \langle\langle c_1, \dots, c_{i-1} \rangle\rangle \leq F_2$ $\begin{matrix} a \mapsto \alpha \\ b \mapsto \beta \end{matrix}$. Now use 4.4 □

Aside

The paper shows that $\exists$ uncountably many f.g. groups up to isomorphism but only countably many f.p. groups.

Example 4.6

0) Finite groups are f.p.

1) $F_n = \langle x_1, \dots, x_n \mid \rangle$

2) $C_n = \langle x \mid x^n \rangle$ (but $\langle x, y \mid x^{14}, y^{21}, x^3 = y^4 \rangle = C_7$

Proposition 4.7

If $N = \langle n_i \mid r_j \rangle, i \in I, j \in J$, $H = \langle h_k \mid s_l \rangle$

then $G = N \rtimes_\varphi H$ has presentation

i.e. $(e, h_k) \cdot (n_i, h_k^{-1}) = (\varphi(h_k)(n_i), e)$

$P = \langle n_i, h_k \mid r_j, s_l, h_k n_i h_k^{-1} = \varphi(h_k)(n_i) \rangle$

Proof

As $h_k n_i = (\text{element of } N) h_k$, any $p \in P$ can be written as $v(n_i) \cdot w(h_k)$ (words in various n_i, h_k).

Define $\theta : P \twoheadrightarrow G$ via $F(\{n_i\} \cup \{h_k\}) \xrightarrow{\text{extend}} G$

If $\theta(p) = e$ then $\theta(\underset{\in N}{v(n_i)})\,\theta(\underset{\in H}{w(h_k)}) = e$ ← θ injective on N, H separately

But $N \cap H = I$ so $v(n_i), w(h_k) = e$ in P.

$\therefore \theta$ is an isomorphism. □

If $G = \langle x_1, \dots, x_n \mid r_1, r_2, \dots \rangle$ and p prime.

Let $\overline{r_j} \in (\mathbb{F}_p)^n$ be the exponential sum vector mod p of r_j. $\overline{r_j} = \begin{pmatrix} \#x_1\text{s} - \#x_1^{-1}\text{s} \\ \#x_2\text{s} - \#x_2^{-1}\text{s} \\ \vdots \end{pmatrix} \bmod p$. $F_n \twoheadrightarrow (\mathbb{F}_p)^n$, $r \mapsto \overline{r}$ is a homomorphism.

Proposition 4.8

If $S = \text{span}\{\bar{r}_i\} \neq (\mathbb{F}_p)^n$ then $G \neq I$.

e.g. $S \neq (\mathbb{F}_p)^n$, could take $a \in S^\perp$
$f : x \mapsto a \cdot x$

Proof

$\exists$ linear $f : (\mathbb{F}_p)^n \twoheadrightarrow \mathbb{F}_p$ with $S \leq \ker(f)$

So $F_n \twoheadrightarrow (\mathbb{F}_p)^n \overset{f}{\twoheadrightarrow} \mathbb{F}_p$ factors through G.

So $G \twoheadrightarrow \mathbb{F}_p$. $\square$

Example 4.9

If # relators < # generators in a finite presentation

then group $G \twoheadrightarrow \mathbb{F}_p \ \forall p$, so it is infinite.

because $\bar{r}_i$ cannot span $\mathbb{F}_p^n$
not enough of them

Proposition 4.10

relators $= \# \bar{r}_i < n$ # generators

If $G = \langle x_i \mid r_k \rangle$, $H = \langle y_j \mid s_l \rangle$

then $G * H = \langle x_i, y_j \mid r_k, s_l \rangle = RHS$

Proof

Let $RHS = G|H \overset{\theta}{\twoheadrightarrow} G * H$ (θ fixes x_i, y_j)

Also, $\exists$ a homomorphism from G into $G|H$ fixing each x_i and also similarly from H. Extend to $\varphi : G * H \to G|H$ by 3.20. Now compose with θ, and extend.

Uniqueness says $\theta\varphi = id$ but φ is injective. $\square$

Nielsen-Schreier

	Free	f.p
Subgroups	✓	✗ (F_2)
Quotients	✗	✗ (4.5)
Extensions	✗	✓

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Topics in Infinite Groups (6)

<u>Theorem 4.11</u> (P. Hall)

If $N \triangleleft G$, then N, G/N f.p $\Rightarrow$ G is also f.p.

<u>Proof</u>

Let $N = \langle x_1, \dots, x_n \mid r_1, \dots, r_m \rangle$

$G/N = \langle y_1, \dots, y_k \mid s_1, \dots, s_k \rangle$

Take $g_1, \dots, g_k$ in G with $g_i N = y_i$. So we have a generating set. $g_1, \dots, g_k, x_1, \dots, x_n$

Now take <u>relations</u> $r_i = e$, $s_i(g_1, \dots, g_k) = t_i(x_1, \dots, x_n) \in N$

$g_j x_i g_j^{-1} = u_{ij}(x_1, \dots, x_n) \in N$.

$g_j^{-1} x_i g_j = v_{ij}(x_1, \dots, x_n) \in N$. $\square$

continues later on.

G a group.

$A, B \leq G$

$\varphi : A \to B$ an isomorphism

HNN extension :

$$G *_\varphi = \frac{G * \langle t \rangle}{\langle\langle tat^{-1} = \varphi(a) \rangle\rangle}$$

---

i) Free Product :

$\{ G_\lambda : \lambda \in \Lambda \}$ an indexed family of groups

ii) Reduced Sequence in $\{G_\lambda\}$:

A finite sequence $g_1, \dots, g_n$ where $g_j \in \coprod_{\lambda \in \Lambda} G_\lambda$

$g_j \neq e$, no successive g_j , g_{j+1} are in the same G_λ.

$\mathcal{R} = \{$reduced sequences$\}$

$\ast_{\lambda \in \Lambda} G_\lambda \leq S(\mathcal{R})$

generated by elements $r_{(g,\lambda)}$

Here, $\lambda \in \Lambda$, $g \in G_\lambda \setminus I$

$r(g, \lambda)\ (g_1 \dots g_n)$

$$= \begin{cases} (g, \lambda)\, g_1 \dots g_n & \text{if } g_1 \notin G_\lambda \\ ((g, \lambda) g_1)\, g_2 \dots g_n & \text{if } g \in G_\lambda \end{cases}$$

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Topics in Infinite Groups (7)

<u>Proof</u> (Theorem 4.11, continued)

Take $\bar{G} = \langle \bar{x}_1, \dots, \bar{x}_n, \bar{g}_1, \dots, \bar{g}_c \rangle$ with the above relations. Then $\bar{G} \xrightarrow{\theta} G$, so let $K = \ker\theta$, and $\bar{N} = \langle \bar{x}_1, \dots, \bar{x}_n \rangle \triangleleft \bar{G}$ by u_{ij}, v_{ij}.

Now restrict θ to $\bar{N}$ where it is an isomorphism. So $K \cap \bar{N} = I$.

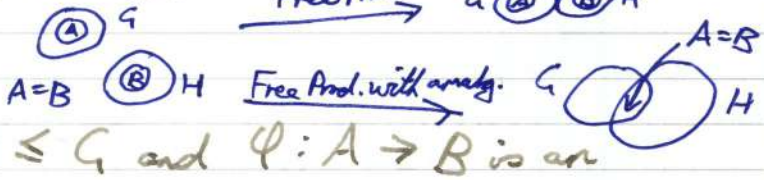
by definition of $\bar{N}$ and G/N

Now, as $\theta(\bar{N}) = N$, we obtain $\theta_0 : \bar{G}/\bar{N} \to G/N$ which is also 1-1 as above, but $\ker\theta_0 = K\bar{N}/\bar{N} \cong K/K\cap\bar{N}$, so $K = I$. $\square$

Free Products with Amalgation and HNN-extension

<u>Definition 4.12</u>

Let G and H be groups with $A \leq G$, $B \leq H$, such that $\varphi : A \to B$ is an <u>isomorphism</u>. The <u>free product with amalgamation</u> is the group $G * H / \langle\langle a = \varphi(a) \ \forall a \in A \rangle\rangle$ ← or a generating set for A

<u>Definition 4.13</u> Idea: 

If G is a group with $A, B \leq G$ and $\varphi : A \to B$ is an isomorphism then the <u>HNN extension</u> $G *_\varphi$ is the group

$G * \langle t \rangle / \langle\langle tat^{-1} = \varphi(a) \rangle\rangle$ (t: stable letter) Idea: Force A, B to be conjugate using new letter t

Note that $ta = \varphi(a)t$, $t^{-1}b = \varphi^{-1}(b)t^{-1}$, so we can move as (resp bs) to the left of ts (t^{-1}s) in $G *_\varphi$.

Choose right transversals T_A and T_B for A and B in G, both including e. A <u>normal form</u> is a sequence $g_0 t^{\varepsilon_1} g_1 t^{\varepsilon_2} \dots t^{\varepsilon_n} g_n$ (for $n \geq 0$, $\varepsilon_i = \pm 1$) such that each $g_i \in G$ with $\varepsilon_i = \pm 1 \Rightarrow g_i \in T_A$ / T_B and there is no subsequence $t^{\varepsilon} e t^{-\varepsilon}$.

allow e no $\emptyset$

Every element of $G * \varphi$ can be put into normal form (work from right to left). – separate large powers of t like $t^3 = tetet$

<u>Theorem 4.14</u> (Normal form for HNNs)

Every element in $G * \varphi$ has a <u>unique</u> normal form.

<u>Proof</u>

set of normal forms?

Define $\rho : G * \varphi \to S(N)$ by:

i) $\rho(g)(g_0 t^{\varepsilon_1} \dots g_n) = g g_0 t^{\varepsilon_1} \dots g_n$ for $g \in G$ $\rho(gh)$

Note that for $g, h \in G$, $\rho(g)\rho(h) = \rho(gh)$ so $\rho(g) \in S(N)$

ii) If $\varepsilon_1 = -1$ and $g_0 \in A$ then

$\rho(t)(g_0 t^{-1} \dots g_n) = (\varphi(g_0) g_1) t^{\varepsilon_2} \dots g_n$

Otherwise $\rho(t)(g_0 t^{\varepsilon_1} \dots g_n) = \varphi(a) t \bar{g}_0 t^{\varepsilon_1} \dots g_n$

where $g_0 = a \bar{g}_0$ for $\bar{g}_0 \in T_A$. $ta = \varphi(a) t$ in $G * \varphi$

This has inverse

$\rho(t^{-1})(g_0 t g_1 \dots g_n) = (\varphi^{-1}(g_0) g_1) t^{\varepsilon_2} \dots g_n$ $(g_0 \in B)$

otherwise $g_0 t^{\varepsilon_1} \dots g_n \mapsto \varphi^{-1}(b) t^{-1} \hat{g}_0 t^{\varepsilon_1} \dots g_n$ $(g_0 = b \hat{g}_0$ for $\hat{g}_0 \in T_B)$

Also $\rho(a) = \rho(t^{-1}) \rho(\varphi(a)) \rho(t)$ so ρ is a well defined homomorphism on $G * \varphi$ and

so ρ is injective hence an isomorphism

$\rho(g_0 t^{\varepsilon_1} \dots g_n)(e) = g_0 t^{\varepsilon_1} \dots g_n$ for normal forms. $\square$

We say that $g_0 t^{\varepsilon_1} \dots g_n$ is <u>reduced</u> if there is no subsequence $t g_i t^{-1}$ for $g_i \in A$ or $t^{-1} g_i t$ for $g_i \in B$.

a "pinch"

reduced sequence need not have $\varepsilon_i = \pm 1$

Corollary 4.15 (Britton's Lemma)

G embeds in $G * \varphi$ by $g \mapsto g$ and if for $n \geq 1$, $g_0 t^{\varepsilon_1} \dots g_n$ is reduced, then it is $\neq e$ in $G * \varphi$.

Proof

so $G \to G * \varphi$ injective

$g\ (\neq e)$, e are both in normal form, so $g \neq e$ in $G * \varphi$ by 4.14. On changing a reduced sequence into a normal one, no $t^{\pm 1}$ cancel so $\xrightarrow[\text{normal form}]{} g_0' t^{\varepsilon_1} \dots g_n'$ $\square$

Corollary 4.16 (Torsion in HNN)

If $r \in G * \varphi$ has finite order, then r is conjugate to some $g \in G$.

Proof

If $r = g_0 t^{\varepsilon_1} \dots g_n$ $(n \geq 1)$ is a reduced sequence and $t^{\varepsilon_n} g_n g_0 t^{\varepsilon_1}$ is not a pinch, then r^k is reduced and so is $\neq e$ by 4.15. Otherwise, replace r by $t^{\varepsilon_n} g_n r (t^{\varepsilon_n} g_n)^{-1} = g t^{\varepsilon_2} \dots t^{\varepsilon_{n-1}} g_{n-1}$ which is either reduced, or in G. Now repeat. shortens word, done by induction $\square$

Inve 2

$\exists$ an infinite group where every element $(\neq e)$ is conjugate.

Proof

Take G to be countably infinite and torsion free (e.g. $\mathbb{Z}$).

Let $\{g_0, g_1, \dots\}$ be an enumeration of non-id elements of G.

We form the following HNN extensions:

$G_1 = \langle G, t_1 \mid t_1 g_0 t^{-1} = g_1 \rangle$ $(\langle g_i \rangle = \mathbb{Z})$

$\leq G_2 = \langle G_1, t_2 \mid t_2 g_0 t_2^{-1} = g_2 \rangle$

G embeds in G_1 by 4.15, so we still have infinite order.

Continue. We obtain $G \leq G_1 \leq \dots$, an ascending sequence.

Now let $\Gamma(G) = \bigcup_n G_n$. Then Γ is countably infinite, and torsion free by 4.16.

Note that any 2 (non-identity) elements of G are conjugate in $\Gamma(G) = \Gamma_1$.

Now set $\Gamma_2 = \Gamma(\Gamma_1), \dots, \Gamma_{m+1} = \Gamma(\Gamma_m)$ and form $\Delta = \bigcup_m \Gamma_m$. Now, $\gamma, \delta^{*e} \in \Delta \Rightarrow \gamma, \delta \in \Gamma_m$, so are conjugate in Γ_{m+1} and so in Δ. $\square$

(D Osin, 2010, $\exists$ f.g. examples. f.p examples, open problem)

For free products with amalgamation, $G *_\varphi H$, we say an element $\cancel{c_1, \dots, c_n}$ $c_1 \dots c_n \in G * H$ is <u>a-reduced</u> if for $n > 1$, no c_i is in A or B. We can turn any $c \in G * H$ into an <u>a-reduced</u> element by "absorbing" elements of A or B.

<u>Corollary 4.17</u>

If $c_1, \dots, c_n$ is a-reduced then it is non-identity in $G *_\varphi H$ ($G, H \hookrightarrow G *_\varphi H$)

<u>Proof</u>

$$A \leq G \xrightarrow{\varphi} B \leq H$$

Let $F = \dfrac{(G * H) * \langle t \rangle}{\langle\langle tat^{-1} = \varphi(a) \rangle\rangle}$, HNN extension of $G * H$.

Let $\Psi : G *_\varphi H \to F$ be defined by $\Psi(g) = tgt^{-1}$, $\Psi(h) = h$.

In $G *_\varphi H$, $a = \varphi(a)$ $\quad a \mapsto tat^{-1}$, $\varphi(a) \mapsto \varphi(a)$ $\quad tat^{-1} = \varphi(a)$ in F

This is well defined: for a-reduced elements, if $c_1 \in A$ then $c_1 \overset{\Psi}{\mapsto} \varphi(a) \neq 1$ (normal form)

Otherwise, $\underset{\in G *_\varphi H}{W} \mapsto$ reduced sequence in F (HNN) so done by 4.15 $\square$

12/01/14

DJS Robinson "A Course in the Theory of Groups"

Topics in Infinite Groups (8)

Definition 5.1

The group G is <u>soluble</u> (solvable) if $\exists$ a sequence $I = G_n \triangleleft G_{n-1} \triangleleft \dots \triangleleft G_1 \triangleleft G_0 = G$, with G_i/G_{i+1} abelian.

Theorem 5.2

This property is preserved by subgroups, quotients and extensions.

Proof

$$\frac{A}{A\cap B} \cong \frac{AB}{B}$$

(subgroups) If $H \leq G$, G soluble, $G_{i+1} \triangleleft G_i \Rightarrow G_{i+1} \cap (H \cap G_i) \triangleleft (H \cap G_i)$

$= (H\cap G_i)\cap G_{i+1}$

and $\dfrac{(H\cap G_i)}{(H\cap G_{i+1})} \cong \dfrac{(H\cap G_i)G_{i+1}}{G_{i+1}} \leq G_i/G_{i+1}$ abelian

(quotients) If $N \triangleleft G$ then $G_{i+1}N/N \triangleleft G_iN/N$ with quotient

$\dfrac{G_iN}{G_{i+1}N} = \dfrac{G_i(G_{i+1}N)}{G_{i+1}N} \cong \dfrac{G_i}{G_i \cap G_{i+1}N} \twoheadleftarrow G_i/G_{i+1}$, abelian.

(extensions) G/N soluble $\Rightarrow$ we have $N \leq G_{i+1} \leq G_i \leq G$, $G_{i+1}/N \triangleleft G_i/N$

So $G_{i+1} \triangleleft G_i$, G_i/G_{i+1} abelian.

Now put this together with $I = N_i \triangleleft N_{i-1} \dots \triangleleft N_0 = N$ with N_i/N_{i+1} abelian. $p: G \to G/N$ $\square$

Take p^{-1} of sequence for G/N to get $G \triangleright \dots \triangleright N$

Then use sequence for N.

Definition 5.3

For any group G the derived series is the sequence of subgroups $G = G^{(0)} \geqslant G^{(1)} \geqslant \dots$ for $G^{(i+1)} = (G^{(i)})'$ <u>derived</u> or commutator subgroup.

$G^{(i+1)}$ is characteristic in $G^{(i)}$.

Proposition 5.4

G is soluble $\Leftrightarrow$ derived series terminates at I.
If so then it has the same length of a smallest series with abelian quotients.

($\Leftarrow$) is clear

Proof

$G^{(i)}/G^{(i+1)} = G^{(i)}/(G^{(i)})'$, so the quotient is abelian.
Now let G_i be as in (5.1) and assume that $G^{(i)} \leq G_i$. As $H \leq G \Rightarrow H' \leq G'$, we have $G^{(i+1)} \leq (G_i)'$
but G_i/G_{i+1} is abelian, so $(G_i)' \leq G_{i+1}$ $\square$
so $G^{(i+1)} \leq G_{i+1}$. Gives result.

So in 5.1, there does exist a series where $G_i \trianglelefteq G$.
and not just $G_i \trianglelefteq G_{i-1}$

We say that G is <u>perfect</u> if $G = G'$. Then $G (\neq I)$ is <u>not</u> soluble, and if $G \twoheadrightarrow Q \twoheadrightarrow A$ abelian, then $A = I$ so Q is also perfect.

e.g. G simple, non-abelian is perfect, like A_5.

Corollary 5.5

If G contains a non-abelian free subgroup, then G is <u>not</u> soluble.

Proof

If G is soluble, $F_2 \leq G$, $F_2/N \cong A_5$ soluble $\divideontimes$ $\square$

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Topics in Infinite Groups (8)

Polycyclic Groups

Definition 5.6

The group G is polycyclic if $\exists$ a chain
$I \triangleleft G_n \triangleleft G_{n-1} \triangleleft \dots \triangleleft G_0 = G$ s.t. G_i/G_{i+1} cyclic.

Theorem 5.7

This is preserved by subgroups, quotients, extensions.

Proof

Exactly the same for as 5.2 with ~~abelian~~ cyclic. $\square$

In fact, if a property P is preserved under subgroups and quotients then 5.2 shows that "poly-P" has all these.
Soluble is "poly-abelian"

Corollary 5.8

F.g. abelian groups are polycyclic.

Proof

Express A as a direct product of cyclic groups. $\square$
via Structure Theorem

Theorem 5.9

G polycyclic $\Leftrightarrow$ G soluble and has max

Proof

($\Rightarrow$) A cyclic group is soluble, has max, and these are preserved by extensions.

has max $\Leftrightarrow$ f.g. (like Noetherian)

f.g, abelian

($\Leftarrow$) Each G_i in (5·1) is f.g. so each G_i/G_{i+1} is polycyclic by 5·8. This is preserved by extensions. $\square$

<u>Corollary</u>

G polycyclic, $H \leq G \Rightarrow H$ is f.p.

<u>Proof</u>

A cyclic group is f.p. and this is preserved by extensions (4.11) so G is f.p. $H \leq G \Rightarrow H$ polycyclic. $\square$

$\Rightarrow H$ is f.p by same reasoning as G

<u>Extended Example 5·11</u>

Let $D \leq \mathbb{Q}$ be the dyadic rationals $\{\frac{n}{2^i} : n \in \mathbb{Z}, i \geq 0\}$. This is not cyclic, hence infinitely generated as $\mathbb{Q}$ is locally cyclic.

a group whose commutator subgroup is abelian. An abelian extension of an abelian group

Let $B = D \rtimes_{\varphi} \overset{\langle t \rangle}{\mathbb{Z}}$ where $\varphi(t)$ is the automorphism $d \mapsto 2d$ of D. Then B is metabelian $\left(\frac{\text{Abelian}}{\text{Abelian}}\right)$ and generated by $1 \in D$ and $t \in \mathbb{Z}$. $(\mathbb{Q},+)$ $(\mathbb{Z})$.

or $d \mapsto d^2$ written multiplicatively

All elements of B have the form $(\frac{n}{2^i}, t^j)$ multiplicative with $(0, t^k)(\frac{n}{2^i}, 1)(0, t^{-k}) = (\frac{2^k n}{2^i}, 1)$.

So $D \leq \langle 1, t \rangle$ by taking $k \geq i$.

So B is polycyclic

What about a finitely presented example?

Let $G = \langle a, b \mid bab^{-1} = a^2 \rangle$.
Then $a \mapsto (1, 1)$, $b \mapsto (0, t)$ extends to a homomorphism $G \to B$ which is surjective.

We show that this is injective.

Now $G/G' \cong \mathbb{Z}$, generated by bG'.
As $ba = a^2 b$ (and $ba^{-1} = a^{-2}b$)
and $ab^{-1} = b^{-1}a^2$ (and $a^{-1}b^{-1} = b^{-1}a^{-2}$)
then given any word in a, b, we can move the powers of b past $a^{\pm 1}$ to the right and b^{-1} to the left.

Thus any $g \in G$ has the form $b^{-m}a^{l}b^{n}$ for $l \in \mathbb{Z}$, $m, n \geq 0$. Now $\theta(g) = (l/2^m, t^{n-m})$, and if $\theta(g) = e_B = (0, 1)$, then $l = 0$, $n = m$
$\Rightarrow \theta$ injective.
$\theta : G \hookrightarrow D \rtimes_\varphi \overset{B}{\overset{\shortparallel}{\mathbb{Z}}}$ G polycyclic as $G \cong B$

This can also be done with matrices. Set $a = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $b = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$. $a, b \in GL_2(\mathbb{Q})$.
Then $b^{-m}a^{l}b^{n} = \begin{pmatrix} 2^{n-m} & l2^{-m} \\ 0 & 1 \end{pmatrix}$
and $bab^{-1} = a^2$.

<u>Proposition 5.12</u>

We have infinite f.p. groups G with f.p $H \leq G$ and $g \in G \setminus H$ s.t. $gHg^{-1} < H$.
Strict Inequality

<u>Proof</u>

Take G as above with $H = \langle a \rangle$. Then $bHb^{-1} = \langle a^2 \rangle$ so $bHb^{-1} < H$ $\square$

<u>Note</u>

In G we have $\ldots < g^2Hg^{-2} < gHg^{-1} < H < g^{-1}Hg < \ldots$
Any example in 5.12 cannot have min. or max. $\square$

17/02/14

Topics in Infinite Groups (9)

Nilpotent Groups

Definition

The group G is <u>nilpotent</u> (nil) if $\exists$ a chain

$I = G_n \leq G_{n-1} \leq \dots \leq G_0 = G$ such that $G_i \trianglelefteq G$

and G_i/G_{i+1} is in the <u>centre</u> of G/G_{i+1} (a <u>central series</u>)

i.e. commutes with everything.

So is abelian

Trivial series $I \leq G$

Note

Ab $\Rightarrow$ Nil $\Rightarrow$ Soluble but $G (\neq I)$ nilpotent $\Rightarrow Z(G) \neq I$

So S_3 is polycyclic but not nilpotent. $\therefore$ nil is <u>not</u> preserved by extensions.

Theorem 5.14

Nil is preserved by subgroups, quotients, and direct products.

Proof

Sub Groups i) As in 5.2), we want $\dfrac{H \cap G_i}{H \cap G_{i+1}}$ is in the centre of $\dfrac{H}{H \cap G_{i+1}} \cong \dfrac{HG_{i+1}}{G_{i+1}}$. The <u>former</u> becomes $\dfrac{(H \cap G_i)G_{i+1}}{G_{i+1}}$ which is in G_i/G_{i+1}, so commutes with G/G_{i+1}, so commutes with $\dfrac{HG_{i+1}}{G_{i+1}}$

Quotient ii) We need $\dfrac{G_i N}{G_{i+1} N}$ in the centre of $\dfrac{G}{G_{i+1} N}$ but

$[x, g] = xgx^{-1}g^{-1} \in G_{i+1}$ for $x \in G_i$, $g \in G$, so

N normal $xn = n_0 x$, $n_0 \in N$

$[xn, g] = xngx^{-1}g^{-1} \in G_{i+1}N$. $\quad xngx^{-1}g^{-1} = n_0 x g x^{-1} n_0^{-1} g^{-1}$ $\quad xgx^{-1} \in G_{i+1}$, $gn_0^{-1}g^{-1} \in N$

Products iii) Given $G_i \leq G$, $H_j \leq H$ as in 5.13, $G_i \times H \trianglelefteq G \times H$

with $\dfrac{G_i \times H}{G_{i+1} \times H}$, so we use

$I \times I = I \times H_m \leq I \times H_{m-1} \leq \dots \leq I \times H = G_n \times H$

$\leq G_{n-1} \times H \leq \dots \leq G \times H$ $\square$

For $H_1, \dots, H_n \leq G$, define $[H_1, H_2] = \langle [h_1, h_2] \rangle$

and $[H_1, \dots, H_n] = [[H_1, \dots, H_{n-1}], H_n]$.

For any G, the <u>lower central series</u> of G is

$$\gamma_1 G = G \geqslant \underset{[G,G]}{\underset{\parallel}{\gamma_2 G}} \geqslant \underset{[[G,G],G]}{\underset{\parallel}{\gamma_3 G}} \geqslant \dots \geqslant \underset{[\gamma_i G, G]}{\underset{\parallel}{\gamma_{i+1} G}} \geqslant \dots$$

<u>Lemma 5.15</u>

G is nilpotent $\Leftrightarrow$ lower central series terminates at I.

<u>Proof</u>

clear for $\gamma_2 G$, follows by induction

($\Leftarrow$) $\gamma_i G$ is characteristic in G and $\gamma_i G / \gamma_{i+1} G$ is in $Z(G/\gamma_{i+1} G)$

$x \in \gamma_i G$, $g \in G$, $[x,g] \in \gamma_{i+1} G$

so we have a central series.

($\Rightarrow$) Assume $\gamma_i G \leq G_{i-1}$ (central series) for $i \geqslant 1$. Then $[\gamma_i G, G] \leq [G_{i-1}, G]$ ($\gamma_{i+1} G$)

which (as a central series) is in G_i. Hence $\gamma_i G \leq G_{i-1}$ $\forall i$

Hence the result.

<u>Theorem 5.16</u> (Baer)

If G is nil and f.g., then $\gamma_i G$ is finitely generated.

<u>Proof</u>

If $\gamma_3 G = I$, and $G = \langle x_1, \dots, x_n \rangle$ is symmetric (closed under inverses) then, as we have

$[y, x] x [y, z] x^{-1} = [y, xz]$ and

$z[x, y] z^{-1} [z, y] = [zx, y]$

clear these xs because $x[y,z]x^{-1}[y,z]^{-1} \in \gamma^3 G$ so $= e$

and similarly for zs

in <u>any</u> group, then in G we get

$[g, xh] = [g, x][g, h]$ and

$[gx, y]$, so any element of $\gamma_2 G$ is a <u>finite</u> product

$[x,y][g,y]$ $\prod [x_j, x_k]$ of the generators

This also shows that for any G, $\gamma_2 G / \gamma_3 G$ is ~~given~~ generated by $\{[x_j, x_k]\}$. Now assume that anything in $\gamma_i G / \gamma_{i+1} G$ is a product of $\{[x_{j_1}, \dots, x_{j_i}] \equiv {}_i[\underline{x}]\}$. (*)

In $\gamma_{i+1} G / \gamma_{i+2} G$, suppose that $g = x_l h$.

Then $[{}_i[\underline{g}], g] = [{}_i[\underline{g}], x_l h]$

$= [{}_i[\underline{g}], x_l][{}_i[\underline{g}], h]$, as $[\overset{i+1}{\dots}]$ is in the centre.

So for $[g_1, \dots, g_{i+1}] \in \gamma_{i+1} G$, we can write it as

$\prod_r [[g_1, \dots, g_i], x_r])\, r$ for $r \in \gamma_{i+2} G$.

$= \left(\prod_r \left(\prod_s \underbrace{[{}_i[\underline{x_s}]\beta, x_r]}_{\text{because } [\beta, x_r][{}_i[\underline{x_s}], x_r]\delta}\right)\right) r'$ for $\beta \in \gamma_{i+1} G$, $r' \in \gamma_{i+2} G$.

, $\delta \in \gamma_{i+2} G$

by assumption (*)

$= \prod_r \prod_s [{}_i[\underline{x_s}], x_r]$ modulo $\gamma_{i+2} G$.

So if $\gamma_{n+1} G = I$, then ($\gamma_n G$ is f.g. and $\gamma_{n-1} G / \gamma_n G$ f.g.) $\Rightarrow \gamma_{n-1} G$ is f.g. and so on. $\square$

<u>Corollary 5.17</u>

All f.g. nilpotent groups G are polycyclic (and have max)

<u>Proof</u>

$\gamma_i G / \gamma_{i+1} G$ is abelian and f.g, so extensions have max and are polycyclic. $\square$

<u>6 Finite Index Subgroups and Virtual Properties</u>

If $H \leq G$ has <u>finite index</u> (i.e. a finite number of cosets) we write $H \leq_f G$ with index $[G : H]$.

Lemma 6.1

i) If $H \leq_f G$, $H \leq J \leq G$, then $H \leq_f J \leq_f G$

ii) If $J \leq_f H \leq_f G$, then $J \leq_f G$ with $[G:J] = [G:H][H:J]$

iii) If $H \leq_f G$, $S \leq G$, then $H \cap S \leq_f S$ with index $\leq [G:H]$, with equality $\Leftrightarrow SH = G$ and it divides $[G:H]$ if $SH \leq G$.

iv) If $H \leq_f G$, ~~then~~ $J \leq_f G$, then $H \cap J \leq_f G$ with $[G : H \cap J] \leq [G:H][G:J]$

Cosets $j_1H, \dots, j_nH$ are a subset of cosets $g_1H, \dots, g_nH$ so $H \leq_f J$

$J = \bigcup_i j_i H$

$g_r J = \bigcup_i g_r j_i H$ so $J \leq_f G$

Proof

i) Check. Like checking the 'tower law' for number fields

Take coset reps.

ii) Suppose $g_1, \dots, g_k$ is a (left) transversal for H in G (collection of coset reps.) and $h_1, \dots, h_l$ for J in H. Then $G = \bigcup_{i,j} g_i h_j J$

Check that if $g_i h_j J = g_{i'} h_{j'} J$, then they are the same coset.

Straightforward

iii) Take $G = g_1 H \cup \dots \cup g_k H$, and throw away any coset $g_i H$ with $(g_i H) \cap S = \phi$ (this happens $\Leftrightarrow SH \neq G$).

Then $g_1(H \cap S), \dots, g_k(H \cap S)$ are disjoint since $g_i H$ are disjoint.

Check that their union U is as required. Replace g_i with $s_i \in S$ as above to form $s_i(H \cap S)$.

i.e. $U = S$

Clear since $g_i H$ cover G.

End of this line gives $[S : H \cap S] \leq [G:H]$

Then $U \subseteq S$ and if $s \in g_i H$ then $s = s_i h$ so $s \in s_i(H \cap S)$

If $SH \leq G$ (a subgroup) then $H \leq SH \leq G$ so use i), ii), and

$[G:H] = [G:SH][SH:H]$

we have shown that $[SH : H] = [S : H \cap S]$

iv) $[G : H \cap J] \overset{(ii)}{=} [G:J][J : H \cap J] \overset{(iii)}{\leq} [G:J][G:H]$ $\square$

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Topics in Infinite Groups (10)

Theorem 6.2 (Poincaré)

A finite intersection of f.i. subgroups has f.i. □

By induction and 6.1(v)

Lemma 6.3

If $[G:H]=k$ then for any $g \in G$, $\exists i$ with $1 \le i \le k$ such that $g^i \in H$. If $H \triangleleft G$ then we can take $i \mid k$ (or even $i = k$).

Proof

Pigeonhole principle

$H, gH, \dots, g^kH$ cannot be distinct cosets so $g^iH = g^jH$ for $0 \le i < j \le k$. Then $g^{j-i} \in H$ for $1 \le j-i \le k$. If $H \triangleleft G$ then gH has order dividing $k = |G/H|$. □

Proposition 6.4

Let $\theta : G \twoheadrightarrow H$ be a surjective homomorphism.

i) If $B \le_f H$ then $\theta^{-1}(B) \le_f G$; in fact $[\overset{G}{\underset{C=\theta^{-1}(B)}{\theta^{-1}(B)}}] = [\overset{H}{\underset{B}{H}}]$

ii) If $A \le_f G$ then $\theta(A) \le_f H$ with $[H:\theta(A)] \mid [G:A]$

Proof

i) We have $H = h_1B \cup \dots \cup h_kB$, and we take g_i with $\theta(g_i) = h_i$ to get a transversal for $C = \theta^{-1}(B)$ in G. For $g \in G$, say $\theta(g) = h_ib$. Then $g_i^{-1}g \in C$ and pullback sends disjoint sets to disjoint sets with $\theta^{-1}(h_iB) = g_iC$.

so $C \le_f G$

---

ii) $\theta^{-1}(\theta(A)) = KA \le_f G$ for $K = \ker\theta$ with $[G:KA] = [H:\theta(A)]$ by i), and $[G:KA] \mid [G:A]$ □

Regular Representation

Any group acts on itself by (left) multiplication.

Now let H be any subgroup and Ω the set of left cosets

of H in G. The (left) <u>regular representation</u> ρ of G on Ω is the action of G given by $\rho(g)(xH) = gxH$.
Note that $Orb(H) = \Omega$ and the stabiliser of $H \in \Omega$ is $H \leq G$.

<u>Lemma 6.5</u> $\rho : G \to \Omega$

$$\ker\rho = \bigcap_{x \in G} xHx^{-1}.$$

<u>Proof</u>

$xH = gxH \ \forall x \in G \iff xgx^{-1} \in H \ \forall x \in G.$ $\square$

<u>Definition 6.6</u>

For $H \leq G$, the <u>core</u> of H in G is $\ker\rho$.

<u>Proposition 6.7</u>

Core $H \lhd G$ is the largest normal subgroup of G that is contained in H.

<u>Proof</u>

because $x^{-1}Nx = N$

If $N \lhd G$ and $N \leq H$, then $x^{-1}Nx \leq H \ \forall x \in G$ $\square$

<u>Theorem 6.8</u> (Useful!)

If $H \leq_f G$ with $[G:H] = n$, then $\exists N \lhd_f G$ with $N \leq H$, and $[G:N] \mid n!$.

<u>Proof</u> $\rho : G \to$ cosets of H

Ω has n elements, so $\rho : G \to S(n)$.

$|G/\ker\rho| = |Im(\rho)|$ which divides $|S(n)| = n!$ $\square$

$N = \ker\rho$

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Topics in Infinite Groups (10)

Theorem 6·9

If G is <u>finitely generated</u> then for any $n \in \mathbb{N}$, $\exists$ only finitely many subgroups of index n in G.

Proof

If $G = \langle g_1, \dots, g_k \rangle$ then $\exists$ only finitely many homomorphisms $\theta : G \to S(n)$ as θ would be determined by $\theta(g_i)$, so only finitely many which are transitive on $\{1, \dots, n\}$, where $\underbrace{\text{Stab}(1) \text{ has index } n}_{\Leftrightarrow \theta \text{ transitive}}$ (Orbit-Stabiliser Theorem). Now, say $H \leq_f G$ with $[G : H] = n$. Then, on ordering Ω as $\{H, \dots\}$, we have that the regular representation of G is a transitive homomorphism from G to $S(n)$ with $\text{Stab}(1) = H$. $\square$

Corollary 6·10

If G is f.g and $H \leq_f G$, then $\exists$ C characteristic in G with $C \leq_f H \leq_f G$.

Proof

Let $C = \bigcap_{\alpha \in \text{Aut}(G)} \alpha(H)$, then H and $\alpha(H)$ have the same index in G by 6·4, with $C \leq_f G$ by $\underbrace{6 \cdot 9}_{\{\alpha(H) \mid \alpha \in \text{Aut}(G)\} \text{ is finite}}$ and 6·2. → finite intersection of f.i. group has f.i.

index of groups under homomorphisms

If $\beta \in \text{Aut}(G)$, then $\beta(C) = \bigcap_{\alpha \in \text{Aut}(G)} \beta\alpha(H)$
$= \bigcap_{\gamma \in \text{Aut}(G)} \gamma(H) = C$ $\square$

Theorem 6·11

If $H \leq_f G$, then G f.g, f.p $\Leftrightarrow$ H f.g, f.p.

Proof

G f.g means that $\exists\, \theta : F_k \twoheadrightarrow G$. Now $H \leq_f G$

$\Rightarrow \Theta^{-1}(H) \leq_f F_K$ by 6·4, so it is F_L by the Nielsen-Schreier Index Theorem. so H is f.g.

Now restrict Θ to $F_L \twoheadrightarrow H$.

Now, suppose $G = F_K/N$ is f.p, so $N = \langle\langle r_1, \dots, r_m \rangle\rangle_{F_K}$

Then $H = F_L/N$ for $N \leq F_L \leq_f F_K$ by the above.

Take a right transversal $t_1, \dots, t_n$ for F_L in F_K where $n = [G:H]$.

By 1·18, N consists of all elements of the form $(g_1 r_{i_1}^{a_1} g_1^{-1}) \cdots (g_j r_{i_j}^{a_j} g_j^{-1})$ for $g_1, \dots, g_j \in F_K$. These need not be in F_L. But as any $g \in F_K$ is $h t_i$ for $h \in F_L$, we do have the normal closure of $\{t_j r_i t_j^{-1} : 1 \leq i \leq m, 1 \leq j \leq n\}$ in F_L is N. $\rightarrow$ Hence H is f.p with these relators

---

Finally, f.g. (resp f.p) groups are preserved by extensions

($\Leftarrow$) (1·29, 4·11) so if H is f.g. (resp f.p.) then take $N \triangleleft G$ with $N \leq_f H$ by 6·8. N is f.g. (resp f.p.) so G is f.g. (resp f.p.) $\square$

$N \leq_f G \Rightarrow G/N$ finite so certainly f.g, f.p

<u>Note</u>

If $G = \langle k \text{ generators} \mid m \text{ relators} \rangle$

then we say that the <u>deficiency</u> def (presentation for G) $= k - m$

($> 0 \Rightarrow G$ infinite)

6·11 has shown that if $\exists H$ with $[G:H] = n$, then H has a presentation with def $= n(k-1-m)+1$.

So def (presentation for H) $- 1 = [G:H]$(def (Presentation for G) $- 1$)

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Topics in Infinite Groups (10)

How do we know a group has <u>proper</u> finite index subgroups?

e.g. If $H \leq_f \mathbb{Q}$ with index n, then $\forall q \in \mathbb{Q}$,

$q = n\left(\frac{q}{n}\right) \in H$ (6.3).

<u>Theorem 6.12</u> (Higman, 1951)

The group $G = \langle a_1, \ldots, a_4 \mid a_1 a_2 a_1^{-1} = a_2^2, \ldots, a_4 a_1 a_4^{-1} = a_1^2 \rangle$ has no proper finite index subgroups.

<u>Proof</u>

If $H <_f G$ then 6.8 gives a non-trivial finite quotient G/N

Note that for $n > 1$, and a prime $p \mid 2^n - 1$, the least prime factor of n is $< p$:

Take r the order of $2 \bmod p$, then $r \mid n, p-1$ $(r \neq 1)$ by Fermat's Little Theorem. Now say n_i is the order of a_i in G/N.

choosing $n = n_1$

Then $a_1^n a_2 a_1^{-n} = a_2^{2^n}$, so $n_2 \mid 2^{n_1} - 1$, and so on.

Let p be the smallest prime that divides $n_1 n_2 n_3 n_4$.

WLOG, $p \mid n_2$. Then n_1 has a smaller prime factor ※ unless $n_1 n_2 n_3 n_4 = 1$ $\square$

So $p \mid n_2 \mid 2^{n_1} - 1$

Then can show n_1 has a smaller prime factor from first part.

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Topics in Infinite Groups (11)

Proposition 6·13

G in 6·12 is infinite.

Proof $G = \langle a_1, a_2, a_3, a_4 \mid a_1 a_2 a_1^{-1} = a_2^2, \dots, a_4 a_1 a_4^{-1} = a_1^2 \rangle$

If $H = \langle x, y \mid yxy^{-1} = x^2 \rangle$, H' the same but with $H' = x' \; y' \quad x'y'x'^{-1} = x'^2$

dashes everywhere (x', y' etc) then x, y have infinite order

by 5·11, so $H *_\varphi H'$, where $\varphi(x) = y'$, is infinite, and is

In depth Example Study

$\langle \underset{(=y')}{x}, y, \underset{\equiv x'}{z} \mid yxy^{-1} = x^2, xzx^{-1} = z^2 \rangle$

i.e. no reduced word is $= e$

Note that y, z freely generate F_2 by (4·17), as a word $w(y, x')$ with powers gathered is a-reduced, and therefore $\neq e$ in $H *_\varphi H'$

Now take 4 copies of H. $H_i = \langle a_i, b_i \mid b_i a_i b_i^{-1} = a_i^2 \rangle$.

Form $H_1 *_\varphi H_2 = \langle a_1, b_1, a_2 \mid b_1 a_1 b_1^{-1} = a_1^2, a_1 a_2 a_1^{-1} = a_2^2 \rangle = K$,

infinite with $\langle b_1, a_2 \rangle \cong F_2$ free.

Similarly, $L = H_3 *_\varphi H_4$ (send $1 \mapsto 3$, $2 \mapsto 4$). $\langle b_3, a_4 \rangle \cong F_2$

Finally, make $K *_\theta L$ for $\theta(b_1) = a_4$, $\theta(a_2) = b_3$.

This is G. contains $\langle b_3, a_4 \rangle \cong F_2$, so infinite $\square$

Virtual Properties

Here, we regard groups H and G as "basically the same" if $H \leq_f G$. We say that a group property $\mathcal{P}$ is "ok" if when G has $\mathcal{P}$ and $H \leq_f G$ then H has $\mathcal{P}$. But what about finite index supergroups: if H has $\mathcal{P}$ and $H \leq_f G$ then G might not have $\mathcal{P}$.

Definition 6·14 The point of an "ok" property is that if $\mathcal{P}$ is not "ok" then "virtually $\mathcal{P}$" is not so useful to define

If a property $\mathcal{P}$ is "ok", we say that a group G is

virtually $\mathcal{P}$ if $\exists H \leq_f G$ where H has $\mathcal{P}$.

By (6.8) this is the same as G is $\mathcal{P}$ by finite. → i.e. N is $\mathcal{P}$, G/N is finite

↳ $H \leq_f G$, $[G:H]=n \Rightarrow \exists N \triangleleft_f G$, $[G:N] \mid n!$ the "Useful Theorem"

For f.g. groups cyclic $\Rightarrow$ abelian $\Rightarrow$ nil $\Rightarrow$ polycyclic $\Rightarrow$ soluble.

trivial; take trivial series $I \triangleleft G$; need f.g; trivial

Example 6.15

i) $\mathbb{Z}\times\mathbb{Z}$ is abelian but not virtually cyclic (Sheet 1)

ii) The following is a nilpotent group that is not virtually abelian.

Let $G = \langle a,b,t \mid ab=ba,\ tat^{-1}=ab,\ tbt^{-1}=b\rangle$ be the semidirect product $\underset{\langle a,b\rangle}{\mathbb{Z}^2} \rtimes_\varphi \underset{\langle t\rangle}{\mathbb{Z}}$ ($\varphi(t^k)(b)=b$, $\varphi(t^k)(a)=ab^k$). Then $b\in Z(G)$ and $G/\langle b\rangle \cong \mathbb{Z}^2$ so G is nilpotent. Now if we have abelian $A \leq_f G$, then $t^i, a^j \in A$ for $i,j>0$ by 6.3. But $t^i a^j t^{-i} = (ab^i)^j = a^j b^{ij} \neq a^j$ in G.

iii) The following is a polycyclic group which is not virtually nilpotent.

$G = \langle a,b,t \mid ab=ba,\ tat^{-1}=a^2b,\ tbt^{-1}=ab\rangle$

could work out as $\underset{\langle a,b\rangle}{\mathbb{Z}^2} \rtimes_\varphi \underset{\langle t\rangle}{\mathbb{Z}}$ for some φ so polycyclic

If $H \leq_f G$ with H nilpotent then take $h \neq e$ in $Z(H)$ and set $h = a^k b^l t^m$. Now $\exists i>0$ with $t^i \in H$ by 6.3 and thus $t^i h t^{-i} = h$. But this changes l if $k\neq 0$ and adds a's if $l \neq 0$.

Word Growth * Non Examinable *

If $G=\langle X\rangle$ for X finite, then the growth function $r_X^{(n)} : \mathbb{N}\to\mathbb{N}$ is $r_X^{(n)} = \#\{g\in G \mid g = w(X) \text{ for } w, \text{ word length} \leq n\}$.

If $S = X \cup X^{-1} \cup \{e\}$ then $r_X^{(n)} = |S^n|$.

We say that finitely generated G has polynomial word growth if either $\exists X$ finite or $\forall$ generating sets, $\exists c,d>0$ s.t. $\forall n$, $r_X(n) \leq cn^d$

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Gromov 1981: G has polynomial word growth

$\iff$ G is virtually nilpotent

**

Virtually Polycyclic Groups

Theorem 6.16

The following are equivalent:

i) G is virtually polycyclic — "— by —"

ii) polycyclic by finite — "extension of a — by a —"

iii) poly ($\mathbb{Z}$) by finite — $G/N \cong Q$

iv) poly ($\mathbb{Z}$ or finite) — say G is an extension of N by Q; G is N by Q; G over Q is N

Proof "Useful" Theorem: $H \leq_f G \Rightarrow \exists N \trianglelefteq_f G$

fairly easy to see

(1) = (2) and (3) $\Rightarrow$ (2) $\Rightarrow$ (4). So we prove (4) $\Rightarrow$ (3):

Suppose that N is $\underbrace{((\ldots(\mathbb{Z} \text{ by } \mathbb{Z})\ldots) \text{ by } \mathbb{Z})}_{P}$ by F) finite and $G/N = \mathbb{Z}$. Now $P \leq_f N$ is finitely generated so by 6.10 we have $P_0 \leq_f N$ and P_0 is characteristic in N with P_0 also poly($\mathbb{Z}$) (preserved by subgroups). Now $P_0 \triangleleft G$ with $G/N \cong (G/P_0)/(N/P_0)$ so G is ($\overset{P_0}{\text{poly } \mathbb{Z}}$) by ($\overset{N/P_0}{\text{finite}}$ by $\overset{G/N}{\mathbb{Z}}$) G/P_0

but the latter (G/P_0) is $\mathbb{Z}$ by finite (using 3.8). $\rightarrow$ If $G_0 \overset{\theta}{\twoheadrightarrow} F_n$ then $G_0 = \ker\theta \rtimes F_n$, $G/P_0 \twoheadrightarrow G/N = \mathbb{Z} = F_1$

So by 1.28, G = (poly $\mathbb{Z}$ by $\mathbb{Z}$) by finite.

G is A by (B by C) $\Rightarrow$ G is (A by B) by C

So in general, we can push all finite factors "to the right", and we can gather, as (P by finite) by finite = P by finite. $\square$

Corollary 6.17

Virtually polycyclic groups are preserved by subgroups, quotients, and extensions. This is the smallest class containing all

finite groups and $\mathbb{Z}$, and all have max and are f.p.

Proof

The property $\mathcal{P}$ = ($\mathbb{Z}$ or finite) is preserved by subgroups and quotients, so poly $\mathcal{P}$ is preserved by subgroups, quotients and extensions by 5.7. Any poly $\mathcal{P}$ group is contained in any class with the above properties, and max, f.p, are preserved by extensions. $\square$

Virtually Soluble Groups

Corollary 6.18

The virtually soluble groups with max. are exactly the virtually polycyclic groups.

Proof

G polycyclic $\Leftrightarrow$ G soluble and has max.

(5.9) and (6.17) $\rightarrow$ G virtually polycyclic $\Leftrightarrow$ G polycyclic by finite $\square$

** Are these all of the groups with max? See last lecture. **

Theorem 6.19

Virtually soluble groups are preserved by subgroups, quotients and extensions.

Proof

i) If $H \leq_f G$ with H soluble and $S \leq G$, then $H \cap S \leq_f S$ by 6.1 iii) but $H \cap S \leq H$ so soluble by 5.2 i), so S is virtually soluble

$\theta : G \twoheadrightarrow H$, $A \leq_f G \Rightarrow \theta(A) \leq_f H$

ii) 6.4 ii) and 5.2 ii) solubility preserved by quotients

Use θ as the projection map onto the quotient

iii) Let G/N, N be virtually soluble, and take $M \triangleleft_f N$ soluble and normal of minimal index. i.e. maximal normal

possible by the "useful theorem"

Then if we have soluble $S \triangleleft_f N$, we have $S \leq M$

as SM is soluble in N, normal, and has finite index by 6.1:). Thus M is the unique normal soluble subgroup of that index, and so is characteristic in N, normal in G.

because automorphisms preserve index, use 6.4

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Proof (continued)

iii) $Q = G/M$, $R = N/M$, $G/N \cong Q/R$.

R finite and has no non-trivial soluble normal subgroups (pull back)
↳ because M had minimal index in N

Now the centraliser $C = C_Q(R) = \bigcap_{r \in R} C_Q(r) \triangleleft_f Q$

by (6.2) and Orb-Stab. → says that each $C_Q(r)$ is f.i. since R is normal in Q and finite.
↳ finite intersection of f.i. groups is f.i.

So the abelian group $C \cap R \triangleleft R$, so $C \cap R = I$. Thus in Q,
↳ because it is a normal soluble subgroup of R.

$\frac{C}{C \cap R} \leftarrow C \cong CR/R \leq Q/R$ so Q/R is virtually soluble.

It is ~~normal~~ soluble because it is finite abelian.

So $G/M = Q$ is too. Now take $H \leq_f G$ with H/M soluble,
→ because R is finite
because $Q = G/M$ is virtually soluble

so H is by 5.2 iii) → solubility preserved by extensions M, H/M both soluble □

Corollary 6.20

Virtually soluble groups are the smallest class preserved by subgroups, quotients and extensions that contains all abelian and finite groups.

$\mathcal{P}$ = (Abelian or finite)?

6.19 says virtually soluble groups are preserved under subgroups, quotients and extensions.

Proof

Like (6.17) but using (6.19) □

As in (5.5), if $F_2 \leq G$, then G is not virtually soluble.

If $S \leq_f G$ and $F_2 \leq G$ then $F_2 \cap S \leq_f F_2$ with

$F_2 \cap S \leq S$ non-abelian, free, so ✱ by 5.5 otherwise we could show that A_5 is soluble

In Sheet 2, we have an example not containing F_2, preserved by subgroups, quotients and extensions.

Finitely generated example $|\cdot 4|$ is not virtually soluble, so no F_2 subgroup.
$S_0(\mathbb{Z})$, finite permutations of integers.

Finitely Presented examples? There are examples but only

5 constructions.

Tits Alternative (72) * * Non Examinable * *

A finitely generated linear group (of $n \times n$ matrices over a field F) or even any linear group with char $= 0$ is either virtually soluble or contains F_2.

7 Maximal (Normal) Subgroups

Definition 7.1

A proper subgroup H is maximal if $H \leq J \leq G$ $\Rightarrow H = J$ or $J = G$.

Example 7.2

$\mathbb{Q}$ has no maximal subgroups: if $M < \mathbb{Q}$ for M maximal then $M \lhd \mathbb{Q}$, and $\mathbb{Q}/M$ has no proper non-trivial subgroups, so is C_p. But $\mathbb{Q}$ has no proper finite index subgroups.

Zorn's Lemma

Poset X, relation $\leq$ (reflexive, transitive, antisymmetric)

Total order: always have $x \leq y$ or $y \leq x$. A subset S of X is a chain if S is totally ordered.

Assume that if every chain S of X has an upper bound in X ($b \in X$ such that $s \leq b$ $\forall s \in S$), then X has maximal element m (if $m \leq x$, then $m = x$).

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Proposition 7.3 (Neumann, 37)

If $H < G$ and $g \in G \setminus H$ then $\exists$ a maximal subgroup M containing H relative to g i.e. $H \leq M < G$ and $g \notin M$ such that if $M \leq L$, $g \notin L$, then $L = M$.

Proof

Poset $= \{ J < G : H \leq J \text{ and } g \notin J \}$ ordered by $\leq$.

Then for a chain $S = \{J_i\}$ we get $\bigcup_i J_i$ a subgroup which contains H but not g, so $\exists$ maximal M. $\square$

Corollary 7.4

Important!

If G is f.g. and $H < G$, then $H \leq M$ for M maximal.

Proof

$G = \langle g_1, \dots, g_k, h_1, \dots, h_l \rangle$ for $g_i \notin H$, $h_j \in H$.

Take M_1 maximal with $H \leq M_1$ relative to g_1.

If $M_1 < L < G$ then $g_1 \in L$ but not all g_i are, so WLOG, $g_2 \notin L$ and if there is no such L, M_1 is maximal so we are already done. Take M_2 maximal with $L \leq M_2$ relative to g_2. This must stop at or before k:

If $g_1, \dots, g_{k-1} \in M_k$ maximal relative to g_k then $g_k \in L$ so $L = G$. so M_k is truly maximal. $\square$

Note

All of this discussion works exactly the same if we replace "subgroup" by "normal subgroup".

Infinite Simple Groups

$G\ (\neq I)$ simple : if $N \triangleleft G$ then $N = I$ or G.

G simple, soluble $\Rightarrow G = C_p$.

Other simple groups are A_n $(n \geq 5)$ and $PSL(n, F)$ for $n > 2$ or $|F| > 3$ (proof does work for infinite fields)

Example 7.5 (Imre 3)

$H = \bigcup_{n \geq 5} A_n \leq S(\mathbb{N})$ as in (1.5 ii) is simple, because if $N \triangleleft H$, then $N \cap A_n \triangleleft A_n$. If $N \neq I$, take k with $N \cap A_k \neq I$, so $A_n \leq N$, $\forall n \geq k$.

H is not f.g.

If G is an infinite simple group then G is not virtually soluble as $S \leq_f G \Rightarrow N \leq S$ with $N \triangleleft_f G$ by (6.8) useful theorem, so $N = S = G$, so G has no finite index subgroups.

In an extremely strong way

Theorem 7.6 (Higman, 51)

Infinite f.g. simple groups exist.

Proof See 6.12. $G = \langle a_1, \dots, a_4 \mid a_1 a_2 a_1^{-1} = a_2^2, \dots, a_4 a_1 a_4^{-1} = a_4^2 \rangle$

no finite index subgroups

Take Higman's G, and $N \triangleleft G$ a maximal normal subgroup, (containing I) so G/N simple (correspondence). But $N \neq G$, $N \not\leq_f G$ so G/N is infinite and f.g. $\square$

by 7.4 adapted to Normal Subgroups

$N \not\leq_f G$

because G is.

because $a_i N a_i^{-1} = N$ but $a_i G a_i^{-1} \lneq G$

What about infinite finitely-presented simple groups?

The Thompson groups F, T, V, finitely presented, infinite

F is not simple, not virtually soluble, no F_2 subgroup.

T, V are simple.

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Topics in Infinite Groups (12)

<u>Open Question</u>

Does there exist an infinite f.p. simple group with no F_2-subgroup.

(Burger and Moses, 1998):

$\exists\, k, l \geq 2$ such that $\Gamma = F_k *_{F_l} F_k$ for $F_l \leq_f F_k$

is f.p., torsion-free, and simple

8 Residual Finite-ness

<u>Definition 8.1</u>

A group G is <u>residually finite</u> (res. fin.) if

$$\bigcap_{H \leq_f G} H = I.$$

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Topics in Infinite Group Theory (13)

Exam

Past Papers : 06/08/14

Friday 30th May. Choose 3 questions from 5. Questions out of 33

Definition 8.1

G is <u>residually finite</u> if $\bigcap_{H \leq_f G} H = I$

Proposition 8.2

The following are equivalent :

i) G is residually finite.

ii) $\bigcap_{N \trianglelefteq_f G} N = I$

iii) $\forall g \in G \setminus \{e\}$, $\exists$ a homomorphism θ onto F, a finite group such that $\theta(g) \neq e$.

iv) $\forall g_1, \dots, g_n \in G \setminus \{e\}$, $\exists \theta \twoheadrightarrow F$, finite, with $\theta(g_i) \neq e$.

Proof

ii) $\Rightarrow$ i) Clear.

i) $\Rightarrow$ ii) By 6.8. "useful theorem"

ii) $\Leftrightarrow$ iii) as for $g \in G \setminus \{e\}$, take $N = \ker\theta$ or $\theta : G \twoheadrightarrow G/N$ where $g \notin N$ ~~take N max rel. to g, N ⊴ G, for example~~

such an N exists by (ii), with $N \trianglelefteq_f G$

ii) $\Rightarrow$ iv) via $\theta : G \twoheadrightarrow G / \bigcap_{i=1}^{n} N_i$ and Poincaré,

take a sufficiently large intersection that it does not contain $g_1, \dots, g_n$, possible by (ii)

whereas iv) $\Rightarrow$ iii) is clear. $\square$

So finite groups and $\mathbb{Z}$ are residually finite, but $\mathbb{Q}$ and the Higman group are not.

An infinite residually-finite ~~subgroup~~ group has infinitely many finite index subgroups (Poincaré) of arbitrarily high index by 6.1 ii): if both $H = \bigcap_{i=1}^{n} H_n$, $J \leq_f G$

$[H : H \cap J][G : H] = [G : H \cap J]$

containment?

then $[G : H] = [G : H \cap J] \Leftrightarrow H \subset J$ so take

$H_{n+1} = J$ with $h \in H \setminus J$, continue, and index $\to \infty$.

* <u>Aside</u> *

Topological groups : Given f.g. group G, we expect the discrete topology. A more interesting topology defines basic open sets to be cosets for N, $N \triangleleft_f G$. This is the <u>profinite</u> topology. Hausdorff $\Leftrightarrow$ G residually finite.

Indiscrete $\Leftrightarrow$ No proper finite index subgroups.

Discrete $\Leftrightarrow$ G finite. * *

<u>Lemma 8.3</u>

If $R_G = \bigcap_{N \triangleleft_f G} N$, then G/R_G is residually finite.

<u>Proof</u>

Normal finite index subgroups of G/R_G are N/R_G for $R_G \leq N \triangleleft_f G$, but for $g \notin R_G$, we have $R_G \leq N \triangleleft_f G$ with $g \notin N$. e.g max rel. g by 7.3 adapted $\square$

Note that if $G \overset{\theta}{\twoheadrightarrow} Q$ is residually finite, then θ factors through G/R_G as $\theta(R_G) \subseteq R_Q$.

<u>Proposition 8.4</u>

i) G residually finite, $H \leq G \Rightarrow H$ residually finite

ii) H residually finite, $H \leq_f G \Rightarrow G$ residually finite

iii) G, H residually finite $\Rightarrow G \times H$ residually finite and if G is <u>finitely generated</u> then $G \rtimes H$ is residually finite.

$J \leq_f G \quad S \leq G$
$\Rightarrow J \cap S \leq_f S$

Proof

i) $R_G = \bigcap_{L \leq_f G} L$ from 8.2 and $R_G \cap H \geq R_H$ by 6.1(iii)

ii) For $H \leq_f G$ we have $L \leq_f H \Rightarrow L \leq_f G$, so $R_G \leq R_H$.
But $R_G \geq R_H$ by i).

iii) For $(g,h) \neq id$, take $\theta_1 : G \twoheadrightarrow F_1$, $\theta_2 : H \twoheadrightarrow F_2$,
$\theta_1(g)$, $\theta_2(h)$ not both e. Then $\theta_1 \times \theta_2 : G \times H \to F_1 \times F_2$,
$(g,h) \mapsto (\theta_1(g), \theta_2(h))$, $(g,h) \not\mapsto (e,e)$.

Uses alternative characterisation of residual finiteness from 8.2

For $G \rtimes H$, we have $\theta : G \rtimes H \twoheadrightarrow H$ with $\theta(gh) = h$, so
$\exists \varphi$ with $\varphi\theta(gh) = \varphi(h) \neq e$ in some finite F, unless $h = e$.
Now, take $L \leq_f G$ with $g \notin L$ ($g \in G \setminus \{e\}$, some g), and
$C \leq_f L$, characteristic in G by 6.10. Then $C \triangleleft G \rtimes H$ so
$CH \leq_f G \rtimes H$. As $G \cap H = I$ inside $G \rtimes H$, we have $g \notin CH \leq_f G \rtimes H$ because $g \in G$, $g \notin C$ □

because H is residually finite

This uses that G is f.g

Corollary 8.5

G virtually polycyclic $\Rightarrow$ G residually finite.

Proof

We have $H \leq_f G$ with H poly($\mathbb{Z}$) by 6.16.

equivalent conditions for being virtually polycyclic

Now suppose that if $M/N = \mathbb{Z}$, N finitely generated and residually finite, then $M \cong N \rtimes \mathbb{Z}$ by 3.8, so M residually finite by 8.4 iii).

about $G \xrightarrow{\theta} F_k$
$\Rightarrow G \cong \ker\theta \rtimes F_k$

Thus H is residually finite and G is too by 8.4 ii) □

Theorem 8.6

Free groups are residually finite.

Proof 1

Recall 3.23, that $F = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$, $G = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \in SL_2(\mathbb{Z})$ freely generate F_2. Given a reduced word $(\neq \emptyset)$ $w \in F_2$, we have $w(F, G) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Take a large prime p, $p > \max(|a-1|, |d-1|, |b|, |c|)$. Then,

$\theta : SL_2(\mathbb{Z}) \to SL_2(\mathbb{F}_p)$ (reduction mod p).

Idea: Use matrix embedding to do F_2

$SL_2(\mathbb{F}_p)$ is finite and $\theta(w(F, G)) \neq e$. So F_2 is residually finite. by alternative characterisation of residual finiteness

Nielson Schreier does F_n, $n \in \mathbb{N}$

Now $F_n \leq F_2$ so these are residually finite. For w $(\neq \emptyset) \in F(X)$, where $X = \{x_i : i \in I\}$, only $x_{i_1}, \dots, x_{i_k}$ appear in w.

Project $F(X) \to F_n$ for the rest

So we have $\theta : F(X) \twoheadrightarrow F_k = F(x_{i_1}, \dots, x_{i_k})$ given by sending the rest of X to e and extending.

Now $\theta(w) \neq e$, so we now have $\varphi : F_k \to$ finite group with $\varphi\theta(w) \neq e$. $\square$

Proof 2

F_2 free on a, b. We will create a reduced word w using $a, b, A = a^{-1}, B = b^{-1}$ (formal inverses)

$$w = {}^{11}A^{10}b^{9}A^{8}B^{6}B^{5}B^{4}A^{3}A^{2}b^{1}A$$

Let $f : F_2 \to S(11)$ (or $n+1$ if length n) be given by

	1	2	3	4	5	6	7	8	9	10	11
a		1		3	4				8		10
b		3				5	6	7	10		

Partial Functions. Each $i \mapsto i \pm 1$.

Is this injective?

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Topics in Infinite Groups (13)

Suppose e.g. $f(b)(9) = 10 = f(b)(11)$?

Then "$B^{10} b^{9}$ <u>reduced</u> ※

$f(a), f(b) \in S(11)$.

Now by the universal property, we can extend to a homomorphism $f : F_2 \to S(11)$, $f(n)(1) = 11$, so $f(w) \neq id$

	1	2	3	4	5	6	7	8	9	10	11
9	11	1	2	3	4	5	6	7	8	9	10
6	11	3	1	2	4	5	6	7	10	8	9

05/03/14 Topics in Infinite Groups (14)

Theorem 8.6

a) G_1, G_2 residually finite $\Rightarrow G_1 * G_2$ residually finite

Proof

First, suppose G_1, G_2 are finite. Given a reduced sequence $g_1 g_2 \dots g_n \in G_1 * G_2$ of length $n \geq 1$, let

$X_n = \{ g \in G_1 * G_2 \mid 0 \leq \text{length}(g) \leq n \}$, a finite set because G_1, G_2 finite

Define an action on X_n via the following:

- If $r \in G_2$ then $r(g_1 \dots g_k) = \begin{cases} r g_1 \dots g_k & \text{if length (RHS)} \leq n \\ g_1 \dots g_k & \text{otherwise (i.e. when } k = n, g_1 \in G_1) \end{cases}$

so $\exists$ a homomorphism $G_2 \to S(X_n)$

- Similarly, $\exists$ a homomorphism $G_1 \to S(X_n)$

We can extend this to a homomorphism $G_1 * G_2 \to S(X_n)$, a finite group. (finite because X_n is a finite set)

Now done ~~by~~ 8.2, definition of res. fin. with homomorphisms.

Now $g_1 g_2 \dots g_n (\emptyset) = g_1 g_2 \dots g_n \neq \emptyset$ so that $g_1 g_2 \dots g_n$ is non-trivial in $S(X_n)$ (acts non-trivially on $\emptyset$)

For $g_1 g_2 \dots g_n$ in general $G_1 * G_2$, choose $N_1 \triangleleft_f G_1$, $N_2 \triangleleft_f G_2$ such that $g_1, \dots, g_n \notin N_1 \cup N_2$ ~~by 8.2 iv)~~ possible as G_1, G_2 res. finite

~~i.e. by choosing "max rel." normal subgroups~~

$G_i \twoheadrightarrow G_i/N_i \hookrightarrow (G_1/N_1) * (G_2/N_2)$.

This extends to $G_1 * G_2 \to (G_1/N_1) * (G_2/N_2)$ and the image of $g_1 \dots g_n$ is reduced so has length n. $\square$

and $g_1, \dots, g_n \notin N_1 \cup N_2$ so do not reduce to identity elements

Hopfian Groups

Definition 8.7

A group G is Hopfian if every injective endomorphism

$\theta : G \twoheadrightarrow G$ is injective.

If not then $G/\ker\theta \cong G$, so G is isomorphic to a proper quotient of itself. Finite groups are Hopfian, $\mathbb{Z}$ too.

The free group on $F(\mathbb{N})$ is <u>not</u> Hopfian as we send $x_1 \mapsto x_1$, $x_{i+1} \mapsto x_i$, but it is residually finite.

<u>Theorem 8.8</u> (Malcev 1940)

A finitely generated, residually finite group G is Hopfian.

<u>Proof</u>

For $\theta : G \twoheadrightarrow G$ and $H \leq_f G$ with index n, $\theta^{-1}(H)$ has index n too, and if $\theta^{-1}(H_1) = \theta^{-1}(H_2)$, then

by 6.4.i) $\theta\theta^{-1}(H_1) = H_1 = H_2$. So the pullback map is injective on {index n subgroups of G}. But this is a finite set by 6.9, so is a permutation. However, $\ker\theta \underbrace{\leq \theta^{-1}(H)}_{\text{as } H \ni e}$

since G is f.g. $\forall H \leq_f G$. So $\ker\theta$ is in $\bigcap_{H \leq_f G} \theta^{-1}(H) = R_G = I$ $\square$

<u>Corollary 8.9</u>

If $g_1, \dots, g_n$ generate the free group F_n, then they <u>freely</u> generate F_n.

$\mathbb{Z}$ residually finite $\Rightarrow \ast_{i=1}^{n} \mathbb{Z} \cong F_n$ residually finite

8.8 $\Rightarrow$ F_n Hopfian as it is also f.g.

<u>Proof</u>

F_n Hopfian, so if $w(g_1, \dots, g_n) = e$ in $F(x_1, \dots, x_n)$.

Then define $\theta : F(g_1, \dots, g_n) \to F(x_1, \dots, x_n)$ by:

(given a symbol g_i), $g_i \mapsto$ (its image on RHS), a reduced word in $x_1, \dots, x_n$. Extend this homomorphism.

This is injective since it hits the generating set.

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Topics in Infinite Groups (14)

F_n is res. fin $\quad$ F_n is Hopfian

Hence this is injective (8.6, 8.8), so $w = \phi$ $\square$

An infinite simple group is Hopfian but not residually finite.

Baumslag - Solitar Groups

Definition 8.10

The Baumslag - Solitar group $B_{m,n} = \langle a, t \mid ta^m t^{-1} = a^n \rangle$ for $m, n \neq 0$. So these include $B_{1,1} = \mathbb{Z} \times \mathbb{Z}$,

$B_{1,-1}$ = (Fundamental Group of the Klein Bottle)

We can also change m, n and keep the group the same:

$B_{m,n} \cong B_{-m,-n}$, $B_{m,n} \cong B_{n,m}$

They are HNN extensions $\overset{\mathbb{Z}}{\langle a \rangle} *_\varphi$ with $\varphi : \langle a^m \rangle \to \langle a^n \rangle$.

Proposition 8.11

$B_{m,n}$ is soluble if $|m| = 1$ or $|n| = 1$ and contains F_2 otherwise.

Proof

If $|m| = 1$ or $|n| = 1$, then WLOG we have $B_{1,n}$, and this is soluble, just as in 5.11 ($B_{1,2}$). Otherwise, $a \notin$ domain or image of φ. So for any reduced word $w(x, y) \in F_2$, we have $w(t, ata^{-1})$ is a reduced sequence in an HNN extension. So this is $\neq e$ by Britton's Lemma. $\square$

so then t, ata^{-1} generate F_2

Theorem 8.12

$B_{2,3}$ is not Hopfian.

Proof

Let $\Theta(t) = t$, $\Theta(a) = a^2$. This is a homomorphism as it

preserves the relation: $\Theta(ta^2t^{-1}) = (ta^2t^{-1})^2$, $\Theta(a^3) = a^6$

Is this injective? Yes, because $tat^{-1}a^{-1} \mapsto ta^2t^{-1}a^{-2} = a$

What about the kernel? $\Theta([tat^{-1}, a]) = [a^3, a^2] = e$

$[tat^{-1}, a] = tat^{-1}atat^{-1}a^{-1}$ is a reduced sequence, so $\neq e$ by Britton's Lemma $\square$

<u>Theorem 8.13</u>

$\exists$ f.g. soluble G which is not f.p.

<u>Proof</u>

Consider $G = B_{2,3}$ and Θ as above. Set $K_i = \ker \Theta^i \trianglelefteq G$ with $K_i < K_{i+1}$. For $y \neq e$, $\Theta(y) = e$, we have $y = \Theta^i(x)$ as Θ^i is injective, so $x \in K_{i+1} \setminus K_i$

not f.p

So $G/(\cup K_i) = Q$ is not f.p. by 4.4, since if so, then $Q = \langle a, t \mid S \rangle$ with $S = \langle\langle s_1, \dots, s_k \rangle\rangle$ in G, so all in K_N (for some N large enough). But $x \in K_{N+1} \setminus K_N$ is e in Q, but not in $S \leq K_N$. $\#$ so Q is not f.p.

Now G' is generated by t^iat^{-i} for $i \in \mathbb{Z}$. But

commutator subgroup of G

$\Theta^i(t^iat^{-i}) = t^ia^{2^i}t^{-i} = a^{3^i}$ $(i \geq 0)$ which commutes with $\Theta^i(a)$. This gives $[t^iat^{-i}, a] \xmapsto{\Theta^i} e$ in Q as it is in $\ker \Theta^i$, as well as $[t^jat^{-j}, t^kat^{-k}]$ by conjugacy (set $j = i+k$). Thus $\Theta(G') = Q'$ is abelian so $Q'' = I$ $\square$

because Θ(generators for G') = generators for Q' and these commute

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Topics in Infinite Groups (15)

9 The Generalised Burnside Problem

Examples of Torsion Groups

Finite groups F, infinite $F \times F \times \dots$, example 1.5.

None are f.g.

1. Generalised Burnside Problem (1902)

Do there exist infinite f.g. torsion groups? (f.g.i.t)

Lemma 9.1

If G is such a group then

This is obvious

i) $G \twoheadrightarrow Q \Rightarrow Q$ finite or Q infinite f.g. torsion

ii) $H \leq_f G \Rightarrow H$ f.g.i.t. by 6.11 $\Leftarrow$ (H f.g. $\Leftrightarrow G$ f.g) for $H \leq_f G$

iii) G is not virtually soluble.

Proof

H inherits these from G

by construction

iii) If H is soluble then H/H' f.g., torsion, abelian, so finite.

Thus H' is f.g.i.t. by ii) and soluble. So continue until $H^{(n)} = I$ ※ □

2. Burnside Problem

If G is f.g. and $\exists K$ such that $\forall g \in G$, $g^K = e$ (bounded torsion) then can G be infinite?

Let $FB(n,K) = \langle x_1, \dots, x_n \mid w^K = e \ \forall w \in F_n \rangle$

Then a group G is n-gen and $g^K = e \ \forall g \in G$

$\Leftrightarrow FB(n,K) \twoheadrightarrow G$, so (2) says:

Do there exist n, K for which $FB(n,K)$ is infinite?

3. Restricted Burnside Problem

Can G in (2) be infinite and residually finite

$\Leftrightarrow$ $FB(n,k)/R$ infinite by 8.3.

1. Golod (1964): yes, $\exists$ infinite f.g. p-groups (every element has order p^k for some k).

Schlage-Puchta (2011):

Let p be some fixed prime.

Definition 9.2

In F_n, the p-value $v_p(w)$ of $e \neq w \in F_n$ is

$\max\{k : w = u^{p^k},\ u \in F_n\}$

Definition 9.3

The p-deficiency (p-def) of a presentation

$\langle x_1, \dots, x_n \mid r_1, r_2, r_3, \dots \rangle$ is $n - \sum_{i=1}^{\infty} \frac{1}{p^{v_p(r_i)}}$

if it converges.

Lemma 9.4

Suppose that F acts on X and $S \triangleleft F$ with $[F:S] = p$.

For $x \in X$, if $\exists\, g \in Stab_F(x) \setminus S$ then $Orb_F(x) = Orb_S(x)$

Proof

because S in F is maximal

We have $S\, Stab_F(x) = F$, so for $f(x) \in Orb_F(x)$,

set $f = s\,t$ then $f(x) = s(x) \in Orb_S(x)$. $\square$

Theorem 9.5

For any prime p, $n \geq 2$, $\exists$ infinite n-generated p-group which is residually finite.

Proof

Suppose that $\langle x_1, \dots, x_n \mid r_1, r_2, \dots \rangle$ is a presentation P with $p\text{-def}(P) \geq 1$ defines $G = F_n/R$

i) $\theta : G \twoheadrightarrow C_p$: We must have $v_p(r_i) = 0$ for at most $n-1$ relators, otherwise $p\text{-def}(P) \leq n-n$. By (4.8), if $\text{span}\{\bar{r_i}\} \neq (\mathbb{F}_p)^n$ then $G \twoheadrightarrow C_p$, but if $v_p(r_i) \geq 1$ then $\bar{r_i} = 0 \in (\mathbb{F}_p)^n$ so $\dim \text{span}\{\bar{r_i}\} < n$. so $\theta : G \twoheadrightarrow C_p$

Thus, let $N = \ker \theta$ and set $N = S/R$ for $R \leq S \triangleleft F_n$ with $[F_n : S] = p$. By the proof of (6.11), N is generated by $p(n-1)+1$ elements and $R = \langle\langle t^j r_i t^{-j} \mid i \in \mathbb{N}, 0 \leq j \leq p-1 \rangle\rangle_S$ give relators, where $\{e, t, \dots, t^{p-1}\}$ is a transversal for S in F_n if $t \notin S$.

What is the p-def ? Take one relator $r = r_i$ in P, and set $k = v_p(r)$, so $r = w^{p^k}$, for $w \in F_n$. We have two cases:

a) If $w \notin S$:

By (9.4) with action conjugacy, $x = r$, $f = w$ (commute), we get

$\text{Conj. Class}_{F_n}(r) = \text{Conj. Class}_S(r)$.

So $\langle\langle \underbrace{r, \dots, t^{p-1} r t^{1-p}}_{\text{conjugate in } S} \rangle\rangle_S = \langle\langle r \rangle\rangle_S$

Now $r = (w^p)^{p^{k-1}}$ for $w^p \in S$

b) If $w \in S$:

$t^j r t^{-j} = (\underbrace{t^j w t^{-j}}_{\in S})^{p^k}$ as $S \triangleleft F_n$

This tells us that $R = \langle\langle r_i \text{ if } v_p(r_i) \text{ in } S = v_p(r_i) - 1 \text{ in } F_n, \; t^j r_i t^{-j}, \; 0 \leq j \leq p-1 \text{ if } v_p \text{ equal} \rangle\rangle_S$

So this presentation Q for N has p-def

$$p(n-1)+1 - \sum_{i=1}^{\infty} \frac{p}{p^{v_p(r_i)}} = p\,(p\text{-def}(P) - 1) + 1$$ ← important factor

ii) If p-def $(P) \geq 1$, then G is infinite:

since we get p-def $(Q) \geq 1$, so $N \twoheadrightarrow C_p$, so repeat to get $G > N_1 > N_2 > \dots$

Now list the non-identity elements of F_n as $\{w_1, w_2, w_3, \dots\}$

and set $P = \langle x_1, \dots, x_n \mid w_1^p, w_2^{p^2}, w_3^{p^3}, \dots \rangle$

then p-def $(P) \geq 1$. So G is infinite and a p-group and finitely generated. What about residual finiteness?

By 8.3, G/R_G residually finite, a p-group, f.g.

Is it infinite? $N_1, N_2, \dots \geq R_G$, so G/R_G is infinite. $\square$

END OF COURSE MATERIAL

2. $FB(n,2)$ abelian, $FB(n, \begin{smallmatrix}3\\4\\6\end{smallmatrix})$ finite (1940-50)

$FB(n,5)$ open. Novikov, Adyan (1970s) $FB(n,k)$ infinite for all odd $k \geq 665$

Ol'Shanski (82): $\forall$ large primes p, $\exists$ f.g. infinite G such that if $1 < H < G$, then $H \cong C_p$, so G has max!

3. No! By Zelmanov (Fields, 1994) and others.

Question

$\exists$? f.p. infinite torsion groups

03/03/14 Topics in Infinite Group Theory (13)

Exam Past Papers : 06/08/14

Friday 30th May. Choose 3 questions from 5. Questions out of 33

Definition 8.1

G is residually finite if $\bigcap_{H \leq_f G} H = I$

Proposition 8.2

The following are equivalent :

i) G is residually finite.

ii) $\bigcap_{N \trianglelefteq_f G} N = I$

iii) $\forall g \in G \setminus \{e\}$, $\exists$ a homomorphism θ onto F, a finite group such that $\theta(g) \neq e$.

iv) $\forall g_1, \dots, g_n \in G \setminus \{e\}$, $\exists \theta \twoheadrightarrow F$, finite, with $\theta(g_i) \neq e$.

Proof

ii) $\Rightarrow$ i) Clear.

i) $\Rightarrow$ ii) By 6.8. "useful theorem"

ii) $\Leftrightarrow$ iii) as for $g \in G \setminus \{e\}$, take $N = \ker \theta$ or $\theta : G \twoheadrightarrow G/N$ where $g \notin N$ → such an N exists by (ii), with $N \trianglelefteq_f G$

ii) $\Rightarrow$ iv) via $\theta : G \twoheadrightarrow G / \bigcap_{i=1}^{n} N_i$ and Poincaré, whereas iv) $\Rightarrow$ iii) is clear. □

take a sufficiently large intersection that it does not contain $g_1, \dots, g_n$, possible by (ii)

So finite groups and $\mathbb{Z}$ are residually finite, but $\mathbb{Q}$ and the Higman group are not.

An infinite residually-finite group has infinitely many finite index subgroups (Poincaré) of arbitrarily high index by 6.1 ii): if both $H = \bigcap_{i=1}^{n} H_n$, $J \leq_f G$

1

$[H : H \cap J][G : H] = [G : H \cap J]$

containment?

then $[G : H] = [G : H \cap J] \Leftrightarrow H \subset J$ so take

$H_{n+1} = J$ with $h \in H \setminus J$, continue, and index $\to \infty$.

* <u>Aside</u> *

Topological groups : Given f.g. group G, we expect the discrete topology. A more interesting topology defines basic open sets to be cosets for N, $N \trianglelefteq_f G$. This is the <u>profinite</u> topology. Hausdorff $\Leftrightarrow$ G residually finite.

Indiscrete $\Leftrightarrow$ No proper finite index subgroups.

<u>Discrete $\Leftrightarrow$ G finite.</u> **

<u>Lemma 8.3</u>

If $R_G = \bigcap_{N \trianglelefteq_f G} N$, then G/R_G is residually finite.

<u>Proof</u>

Normal finite index subgroups of G/R_G are N/R_G for $R_G \leq N \trianglelefteq_f G$, but for $g \notin R_G$, we have $R_G \leq N \trianglelefteq_f G$ with $g \notin N$. e.g max rel. g by 7.3 adapted $\square$

Note that if $G \overset{\theta}{\twoheadrightarrow} Q$ is residually finite, then θ factors through G/R_G as $\theta(R_G) \subseteq R_Q$.

<u>Proposition 8.4</u>

i) G residually finite, $H \leq G \Rightarrow H$ residually finite

ii) H residually finite, $H \leq_f G \Rightarrow G$ residually finite

iii) G, H residually finite $\Rightarrow G \times H$ residually finite and if G is <u>finitely generated</u> then $G \rtimes H$ is residually finite.

$J \leq_f G \quad S \leq G$
$\Rightarrow J \cap S \leq_f S$

Proof

i) $R_G = \bigcap_{L \leq_f G} L$ from 8·2 and $R_G \cap H \geq R_H$ by 6·1(iii)

ii) For $H \leq_f G$ we have $L \leq_f H \Rightarrow L \leq_f G$, so $R_G \leq R_H$.

But $R_G \geq R_H$ by i).

iii) For $(g,h) \neq id$, take $\theta_1 : G \twoheadrightarrow F_1$, $\theta_2 : H \twoheadrightarrow F_2$,

Uses alternative characterisation of residual finiteness from 8·2

$\theta_1(g)$, $\theta_2(h)$ not both e. Then $\theta_1 \times \theta_2 : G * H \to F_1 \times F_2$,

$(g,h) \mapsto (\theta_1(g), \theta_2(h))$, $(g,h) \not\mapsto (e,e)$.

For $G \rtimes H$, we have $\theta : G \rtimes H \twoheadrightarrow H$ with $\theta(gh) = h$, so

$\exists \varphi$ with $\varphi\theta(gh) = \varphi(h) \neq e$ in some finite F, unless $h = e$.

because H is residually finite

Now, take $L \leq_f G$ with $g \notin L$ ($g \in G \setminus \{e\}$, some g), and

$C \leq_f L$, characteristic in G by 6·10. Then $C \triangleleft G \rtimes H$ so

This uses that G is f.g

$CH \leq_f G \rtimes H$. As $G \cap H = I$ inside $G \rtimes H$, we have $g \notin CH \leq_f G \rtimes H$ because $g \in G$, $g \notin C$ □

Corollary 8·5

G virtually polycyclic $\Rightarrow$ G residually finite.

Proof

We have $H \leq_f G$ with H poly($\mathbb{Z}$) by 6·16. equivalent conditions for being virtually polycyclic

Now suppose that if $M/N = \mathbb{Z}$, N finitely generated

and residually finite, then $M \cong N \rtimes \mathbb{Z}$ by 3·8, so

about $G \xrightarrow{\theta} F_k$
$\Rightarrow G \cong \ker\theta \rtimes F_k$

M residually finite by 8·4 iii).

Thus H is residually finite and G is too by 8·4 ii) □

Theorem 8·6

Free groups are residually finite.

Proof 1

Recall 3.23, that $F = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$, $G = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \in SL_2(\mathbb{Z})$ freely generate F_2. Given a reduced word $(\neq \emptyset)$ $w \in F_2$, we have $w(F, G) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Take a large prime p, $p > \max(|a-1|, |d-1|, |b|, |c|)$. Then,

$\theta : SL_2(\mathbb{Z}) \to SL_2(\mathbb{F}_p)$ (reduction mod p).

Idea: Use matrix embedding to do F_2

$SL_2(\mathbb{F}_p)$ is finite and $\theta(w(F, G)) \neq e$. So F_2 is residually finite. by alternative characterisation of residual finiteness

Nielson Schreier does F_n, $n \in \mathbb{N}$

Now $F_n \leq F_2$ so these are residually finite. For w $(\neq \emptyset) \in F(X)$, where $X = \{x_i : i \in I\}$, only $x_{i_1}, \dots, x_{i_k}$ appear in w.

Project $F(X) \twoheadrightarrow F_n$ for the rest

So we have $\theta : F(X) \twoheadrightarrow F_k = F(x_{i_1}, \dots, x_{i_k})$ given by sending the rest of X to e and extending.

Now $\theta(w) \neq e$, so we now have $\varphi : F_k \to$ finite group with $\varphi\theta(w) \neq e$. $\square$

Proof 2

F_2 free on a, b. We will create a reduced word w using $a, b, A = a^{-1}, B = b^{-1}$ (formal inverses)

$$w = {}^{11}A^{10}b^{9}A^{8}B^{6}B^{5}B^{4}A^{3}A^{2}b^{1}A$$

Let $f : F_2 \to S(11)$ (or $n+1$ if length n) be given by

	1	2	3	4	5	6	7	8	9	10	11
a		1		3	4				8		10
b	1	3				5	6	7	10		

Partial Functions. Each $i \mapsto i \pm 1$.

Is this injective?

Suppose e.g. $f(b)(9) = 10 = f(b)(11)$?

Then "$B^{10} b^{9}$ <u>reduced</u> ※

$f(a), f(b) \in S(11)$.

Now by the universal property, we can extend to a homomorphism $f : F_2 \to S(11)$, $f(w)(1) = 11$, so $f(w) \neq id$

	1	2	3	4	5	6	7	8	9	10	11
a	11	1	2	3	4	5	6	7	8	9	10
b	11	3	1	2	4	5	6	7	10	8	9

05/03/14 Topics in Infinite Groups (14)

Theorem 8.6

a) G_1, G_2 residually finite $\Rightarrow G_1 * G_2$ residually finite

Proof

First, suppose G_1, G_2 are finite. Given a reduced sequence $g_1 g_2 \cdots g_n \in G_1 * G_2$ of length $n \geq 1$, let

$X_n = \{ g \in G_1 * G_2 \mid 0 \leq \text{length}(g) \leq n \}$, a finite set. because G_1, G_2 finite

Define an action on X_n via the following:

- If $r \in G_2$ then $r(g_1 \cdots g_k) = \begin{cases} r g_1 \cdots g_k & \text{if length (RHS)} \leq n \\ g_1 \cdots g_k & \text{otherwise (i.e. when } k = n, g_1 \in G_1) \end{cases}$

so $\exists$ a homomorphism $G_2 \to S(X_n)$

- Similarly, $\exists$ a homomorphism $G_1 \to S(X_n)$

We can extend this to a homomorphism $G_1 * G_2 \to S(X_n)$, a finite group. (finite because X_n is a finite set)

Now done by 8.2, definition of res. fin. with homomorphisms.

Now $g_1 g_2 \cdots g_n (\emptyset) = g_1 g_2 \cdots g_n \neq \emptyset$ so that $g_1 g_2 \cdots g_n$ is non-trivial in $S(X_n)$ (acts non-trivially on $\emptyset$)

For $g_1 g_2 \cdots g_n$ in general $G_1 * G_2$, choose $N_1 \triangleleft_f G_1$, ~~i.e. by choosing "max rel." normal subgroups~~ $N_2 \triangleleft_f G_2$ such that $g_1, \ldots, g_n \notin N_1 \cup N_2$ ~~by 8.2 iv)~~, possible as G_1, G_2 res. finite

$G_i \twoheadrightarrow G_i/N_i \hookrightarrow (G_1/N_1) * (G_2/N_2)$.

This extends to $G_1 * G_2 \to (G_1/N_1) * (G_2/N_2)$ and the image of $g_1 \cdots g_n$ is reduced so has length n. $\square$

and $g_1, \ldots, g_n \notin N_1 \cup N_2$ so do not reduce to identity elements

Hopfian Groups

Definition 8.7

A group G is Hopfian if every injective endomorphism

$\theta : G \twoheadrightarrow G$ is injective.

If not then $G/\ker\theta \cong G$, so G is isomorphic to a proper quotient of itself. Finite groups are Hopfian, $\mathbb{Z}$ too.

The free group on $F(N)$ is <u>not</u> Hopfian as we send $x_1 \mapsto x_1$, $x_{i+1} \mapsto x_i$, but it is residually finite.

<u>Theorem 8.8</u> (Malcev 1940)

A finitely generated, residually finite group G is Hopfian.

<u>Proof</u>

For $\theta : G \twoheadrightarrow G$ and $H \leq_f G$ with index n, $\theta^{-1}(H)$ has index n too, and if $\theta^{-1}(H_1) = \theta^{-1}(H_2)$, then

by 6.4.i) $\theta\theta^{-1}(H_1) = H_1 = H_2$. So the pullback map is injective on {index n subgroups of G}. But this is a finite set by 6.9, so is a permutation. However, $\ker\theta \underbrace{\leq}_{\text{as } H \ni e} \theta^{-1}(H)$

since G is f.g. $\forall H \leq_f G$. So $\ker\theta$ is in $\bigcap_{H \leq_f G} \theta^{-1}(H) = R_G = I$ $\square$

<u>Corollary 8.9</u>

If $g_1, \dots, g_n$ generate the free group F_n, then they <u>freely</u> generate F_n.

<u>Proof</u>

$\mathbb{Z}$ residually finite $\Rightarrow \ast_{i=1}^{n} \mathbb{Z} \cong F_n$ residually finite
8.8 $\Rightarrow F_n$ Hopfian as it is also f.g.

F_n Hopfian, so if $w(g_1, \dots, g_n) = e$ in $F(x_1, \dots, x_n)$.

Then define $\theta : F(g_1, \dots, g_n) \to F(x_1, \dots, x_n)$ by:

(given a symbol g_i), $g_i \mapsto$ (its image on RHS), a reduced word in $x_1, \dots, x_n$. Extend this homomorphism.

This is injective since it hits the generating set.

05/03/14

Topics in Infinite Groups (14)

F_n is res. fin $\quad$ F_n is Hopfian

Hence this is injective $(8.6, 8.8)$, so $w = \phi$ $\square$

An infinite simple group is Hopfian but not residually finite.

<u>Baumslag - Solitar Groups</u>

<u>Definition 8.10</u>

The <u>Baumslag - Solitar</u> group $B_{m,n} = \langle a, t \mid ta^m t^{-1} = a^n \rangle$ for $m, n \neq 0$. So these include $B_{1,1} = \mathbb{Z} \times \mathbb{Z}$,

$B_{1,-1}$ = (Fundamental Group of the Klein Bottle)

We can also change m, n and keep the group the same:

$B_{m,n} \cong B_{-m,-n}$, $B_{m,n} \cong B_{n,m}$

They are HNN extensions $\langle a \rangle *_\varphi$ with $\varphi : \langle a^m \rangle \to \langle a^n \rangle$.

<u>Proposition 8.11</u>

$B_{m,n}$ is soluble if $|m| = 1$ or $|n| = 1$ and contains F_2 otherwise.

<u>Proof</u>

If $|m| = 1$ or $|n| = 1$, then WLOG we have $B_{1,n}$, and this is soluble, just as in 5.11 ($B_{1,2}$). Otherwise, $a \notin$ domain or image of φ. So for any reduced word $w(x, y) \in F_2$, we have $w(t, ata^{-1})$ is a reduced sequence in an HNN extension. So this is $\neq e$ by Britton's Lemma. $\square$

so then t, ata^{-1} generate F_2

<u>Theorem 8.12</u>

$B_{2,3}$ is not Hopfian.

<u>Proof</u>

Let $\Theta(t) = t$, $\Theta(a) = a^2$. This is a homomorphism as it

preserves the relation: $\theta(ta^2t^{-1}) = (ta^2t^{-1})^2$, $\theta(a^3) = a^6$

Is this surjective? Yes, because $tat^{-1}a^{-1} \mapsto ta^2t^{-1}a^{-2} = a$

What about the kernel? $\theta([tat^{-1}, a]) = [a^3, a^2] = e$

$[tat^{-1}, a] = tat^{-1}atat^{-1}a^{-1}$ is a reduced sequence, so $\neq e$ by Britton's Lemma. $\square$

<u>Theorem 8.13</u>

$\exists$ f.g. soluble G which is not f.p.

<u>Proof</u>

Consider $G = B_{2,3}$ and θ as above. Set $K_i = \ker \theta^i \trianglelefteq G$ with $K_i < K_{i+1}$. For $y \neq e$, $\theta(y) = e$, we have $y = \theta^i(x)$ as θ^i is surjective, so $x \in K_{i+1} \setminus K_i$

not f.p

So $G/(\cup K_i) = Q$ is not f.p. by 4.4, since if so, then $Q = \langle a, t \mid S \rangle$ with $S = \langle\langle s_1, \dots, s_k \rangle\rangle$ in G, so all in K_N (for some N large enough). But $x \in K_{N+1} \setminus K_N$ is e in Q, but not in $S \leq K_N$. $\#$ so Q is not f.p. $\blacksquare$

Now G' is generated by t^iat^{-i} for $i \in \mathbb{Z}$. But

commutator subgroup of G

$\theta^i(t^iat^{-i}) = t^ia^{2^i}t^{-i} = a^{3^i}$ $(i \geq 0)$ which commutes with $\theta^i(a)$. This gives $[t^iat^{-i}, a] \xmapsto{\theta^i} e$ in Q as it is in $\ker \theta^i$, as well as $[t^jat^{-j}, t^kat^{-k}]$ by conjugacy (set $j = i+k$). Thus $\theta(G') = Q'$ is abelian so $Q'' = I$ $\square$

because θ(generators for G') = generators for Q' and these commute

10/03/14 Topics in Infinite Groups (15)

9 The Generalised Burnside Problem

Examples of Torsion Groups

Finite groups F, infinite $F \times F \times \ldots$, example 1.5.

None are f.g.

1. Generalised Burnside Problem (1902)

Do there exist infinite f.g. torsion groups? (f.g.i.t)

Lemma 9.1

If G is such a group then

This is obvious

i) $G \twoheadrightarrow Q \Rightarrow Q$ finite or Q infinite f.g. torsion

ii) $H \leq_f G \Rightarrow H$ f.g.i.t. by 6.11 $\Leftarrow$ (H f.g. $\Leftrightarrow G$ f.g) for $H \leq_f G$

iii) G is not virtually soluble.

Proof

H inherits these from G

by construction

iii) If H is soluble then H/H' f.g., torsion, abelian, so finite.

Thus H' is f.g.i.t. by ii) and soluble. So continue until $H^{(n)} = I$ ※ □

2. Burnside Problem

If G is f.g. and $\exists K$ such that $\forall g \in G$, $g^K = e$ (bounded torsion) then can G be infinite?

Let $FB(n,K) = \langle x_1, \ldots, x_n \mid w^K = e \ \forall w \in F_n \rangle$

Then a group G is n-gen and $g^K = e \ \forall g \in G$

$\Leftrightarrow FB(n,K) \twoheadrightarrow G$, so (2) says:

Do there exist n, K for which $FB(n,K)$ is infinite?

3. Restricted Burnside Problem

Can G in (2) be infinite and residually finite

$\Leftrightarrow$ $FB(n,k)/R$ infinite by 8.3.

1. Golod (1964): yes, $\exists$ infinite f.g. p-groups (every element has order p^k for some k).

Schlage-Puchta (2011):

Let p be some fixed prime.

Definition 9.2

In F_n, the <u>p-value</u> $v_p(w)$ of $e \neq w \in F_n$ is

$\max\{k : w = u^{p^k},\ u \in F_n\}$

Definition 9.3

The <u>p-deficiency</u> (p-def) of a presentation

$\langle x_1, \dots, x_n \mid r_1, r_2, r_3, \dots \rangle$ is $n - \sum_{i=1}^{\infty} \frac{1}{p^{v_p(r_i)}}$

if it converges.

Lemma 9.4

Suppose that F acts on X and $S \triangleleft F$ with $[F:S] = p$.

For $x \in X$, if $\exists\, g \in Stab_F(x) \setminus S$ then $Orb_F(x) = Orb_S(x)$.

Proof

because S in F is maximal

We have $S\, Stab_F(x) = F$, so for $f(x) \in Orb_F(x)$,

set $f = s\,t$ then $f(x) = s(x) \in Orb_S(x)$. $\square$

Theorem 9.5

For any prime p, $n \geq 2$, $\exists$ infinite n-generated p-group which is residually finite.

Proof

Suppose that $\langle x_1, \dots, x_n \mid r_1, r_2, \dots \rangle$ is a presentation P with $p\text{-def}(P) \geq 1$ defines $G = F_n/R$

i) $\theta : G \twoheadrightarrow C_p$: We must have $v_p(r_i) = 0$ for at most $n-1$ relators, otherwise $p\text{-def}(P) \leq n-n$. By (4.8), if $\text{span}\{\bar{r_i}\} \neq (\mathbb{F}_p)^n$ then $G \twoheadrightarrow C_p$, but if $v_p(r_i) \geq 1$ then $\bar{r_i} = 0 \in (\mathbb{F}_p)^n$ so $\dim \text{span}\{\bar{r_i}\} < n$. so $\theta : G \twoheadrightarrow C_p$

Thus, let $N = \ker \theta$ and set $N = S/R$ for $R \leq S \triangleleft F_n$ with $[F_n : S] = p$. By the proof of (6.11), N is generated by $p(n-1)+1$ elements and $R = \langle\langle t^j r_i t^{-j} \mid i \in \mathbb{N}, 0 \leq j \leq p-1 \rangle\rangle_S$ give relators, where $\{e, t, \dots, t^{p-1}\}$ is a transversal for S in F_n if $t \notin S$.

What is the p-def ? Take one relator $r = r_i$ in P, and set $k = v_p(r)$, so $r = w^{p^k}$, for $w \in F_n$. We have two cases:

a) If $w \notin S$:

By (9.4) with action conjugacy, $x = r$, $f = w$ (commute), we get

$\text{Conj. Class}_{F_n}(r) = \text{Conj. Class}_S(r)$.

So $\langle\langle \underbrace{r, \dots, t^{p-1} r t^{1-p}}_{\text{conjugate in } S} \rangle\rangle_S = \langle\langle r \rangle\rangle_S$

Now $r = (w^p)^{p^{k-1}}$ for $w^p \in S$

b) If $w \in S$:

$t^j r t^{-j} = (\underbrace{t^j w t^{-j}}_{\in S})^{p^k}$ as $S \triangleleft F_n$

This tells us that $R = \langle\langle r_i \text{ if } \overset{v_p(r_i)}{\text{in } S} = \overset{v_p(r_i)-1}{\text{in } F_n}, \; t^j r_i t^{-j}, 0 \leq j \leq p-1 \text{ if } v_p \text{ equal} \rangle\rangle_S$

So this presentation Q for N has p-def

$$p(n-1)+1 - \sum_{i=1}^{\infty} p^{\frac{p}{v_p(r_i)}} = p\,(p\text{-def}(P) - 1) + 1$$

← important factor

ii) If p-def $(P) \geq 1$, then G is infinite:

since we get p-def $(Q) \geq 1$, so $N \twoheadrightarrow C_p$, so repeat to get $G > N_1 > N_2 > \dots$

Now list the non-identity elements of F_n as $\{w_1, w_2, w_3, \dots\}$ and set $P = \langle x_1, \dots, x_n \mid w_1^p, w_2^{p^2}, w_3^{p^3}, \dots \rangle$

then p-def $(P) \geq 1$. So G is infinite and a p-group and finitely generated. What about residual finiteness?

By 8.3, G/R_G residually finite, a p-group, f.g.

Is it infinite? $N_1, N_2, \dots \geq R_G$, so G/R_G is infinite. $\square$

END OF COURSE MATERIAL

2. $FB(n,2)$ abelian, $FB(n, \frac{3}{4}, 6)$ finite (1940-50)

$FB(n,5)$ open. Novikov, Adyan (1970s) $FB(n,k)$ infinite for all odd $k \geq 665$

Ol'Shanski (82): $\forall$ large primes p, $\exists$ f.g. infinite G such that if $I < H < G$, then $H \cong C_p$, so G has max!

3. No! By Zelmanov (Fields, 1994) and others.

Question

$\exists$? f.p. infinite torsion groups