Topics in Infinite Groups

1 Introduction

Example 1.1

i) \( X \) a set, \( S(X) \) the permutation group of \( X \)

\[
\text{\ldots ii) } \text{A field, } \text{GL}_n(F) \text{ a group under multiplication}
\]

SUBGROUPS:

Proposition 1.2

\( H \subseteq G \) is a subgroup of \( G \) \( \iff \) \( H \neq \emptyset \) and \( \forall a, b \in H, \ ab^{-1} \in H \). We write \( H \leq G \) and \( H < G \) if \( H \neq G \).

\( \text{proper subgroup} \)

\( I = \{e\}, \text{ trivial subgroup} \)

Proposition 1.3

(i) \( L \leq H \) and \( H \leq G \) \( \Rightarrow \) \( L \leq G \).

(ii) If \( H_i \leq G \ \forall i \in I \) then \( \cap_{i \in I} H_i \leq G \).

For \( H_1, H_2 \leq G, \ H_1 \cup H_2 \) is not generally a subgroup. But...

Provision 1.4 (Ascending sequence of subgroups)

If \( H_1 \leq H_2 \leq H_3 \leq \ldots \leq G \) (which means \( H_n \leq G \) and \( H_n \leq H_{n+1} \ \forall n \)) then \( \bigcup_{n=1}^{\infty} H_n \leq G \).

If we have groups \( G_n \ (n \in \mathbb{N}) \) for which \( G_n \leq G_{n+1} \) and we form the set \( X = \bigcup_{n=1}^{\infty} G_n \) then \( X \) is a group.

Example 1.5

\[
\text{i) } S^1 \text{ under multiplication, } H_n = \{ e^{2\pi i k/n} : 1 \leq k \leq 2^n \}, \bigcup_{n=1}^{\infty} H_n \text{ (quasicylic)}
\]

\[
\text{\ldots ii) } G = S(\mathbb{Z}), \ H_n = \{ \text{perms of } \{-n, \ldots, n\}, \text{fix the rest} \} \leq S_{2n+1} \leq H_{n+1}
\]

Generators: Let \( X \subseteq G \).

\( \bigcup H_n = S_0(\mathbb{Z}), \text{ finite support} \)
Definition 1.6
The subgroup \( \langle X \rangle \) generated by \( X \) is \( \cap H \) over all \( H \) with \( X \subseteq H \leq G \). It's the smallest subgroup of \( G \) containing \( X \). We write \( \langle x_1, \ldots, x_k \rangle, \langle X, Y \rangle, \langle X_i : i \in I \rangle \) etc.

Definition 1.7
\( G \) is finitely generated \((\text{f} \text{.} \text{g}.)\) if \( G = \langle g_1, \ldots, g_k \rangle \). Otherwise \( G \) is infinitely generated \((\text{i} \text{.} \text{g}.)\).

Example 1.8
\( G = \langle g \rangle \) means \( G \) is cyclic. Either \( G = \mathbb{Z} \) or, if \( G \) has order \( n \), we write \( G = C_n \).

\[ \text{If } G \text{ has no non-trivial subgroups, } G = \langle e \rangle \text{ or } \mathbb{Z}. \]

Proposition 1.9
If \( X \subseteq G \) then the elements of \( \langle X \rangle \) are \( \langle e \text{ and } x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k} \rangle \) for \( n_1, \ldots, n_k \in \mathbb{Z} \)

and \( x_1, \ldots, x_k \in X \) (but not necessarily distinct).

So \( G \) finite \( \Rightarrow \) \( G \) finitely generated \( \Rightarrow \) \( G \) countable.

If we have an ascending sequence of subgroups \( H_1 \leq H_2 \leq \ldots \leq G \) then either

\[ \exists N \text{ such that } H_N = H_{N+1} \forall n \in \mathbb{N} \] (terminates)

or (on throwing away duplicates) \( H_1 < H_2 < \ldots \) (strictly ascending).

Definition 1.10
\( G \) has \( \text{max} \) (satisfies the maximal condition) if every ascending sequence of subgroups terminates.

Theorem 1.11 \( G \) has \( \text{max} \) \( \Leftrightarrow \) \( H \) is finitely generated \( \forall H \leq G \).

Warning! We can have \( G \) finitely generated but \( H < G \) with \( H \) infinitely generated. We will see examples later; indeed it can be argued that many finitely generated groups do not have \( \text{max} \).

**COSETS:** If \( H \leq G \) then the left cosets of \( H \) in \( G \) are the sets \( gH = \{ gh : h \in H \} \) for each \( g \in G \).
Proposition 1.12 (Lagrange for infinite groups)
The left cosets of $H$ in $G$ form a partition of $G$ and any left coset is in
bijection with $H$.

Note: We have right cosets $Hg$ in bijection with $H$ too. Also there is a
bijection from the set of left cosets to the set of right cosets given by $gH$
goes to $Hg^{-1}$ (not to $Hg$).

Definition 1.13
(i) The index of $H$ in $G$ is the cardinality of the set of left (or right) cosets,
(ii) A left (or right) transversal is a choice of left (or right) coset repre-
sentatives, with exactly one for each coset.

NORMAL SUBGROUPS: For $g, x \in G$ and $H \leq G$ the conjugate of $x$
by $g$ is $gxg^{-1} \in G$ and the conjugate of $H$ by $g$ is $gHg^{-1} \leq G$. Conjugacy
is an equivalence relation.

Definition 1.14
The subgroup $N$ is normal in $G$ (we write $N \trianglelefteq G$) if:
(i) $gN = Ng \ \forall g \in G$ \iff 
(ii) $gNg^{-1} = N \ \forall g \in G$ \iff 
(iii) $gNg^{-1} \leq N \ \forall g \in G$ \iff 
(iv) $N$ is a union of conjugacy classes of $G$.

Examples:
(i) $I \leq G$ and $G \leq G$.
(ii) If $G$ is abelian then $H \leq G$ for all subgroups $H$ (but not the converse).
(iii) If $[G : H] = 2$ then $H \leq G$.

Proposition 1.15 (cf. Proposition 1.3)
(i) If $N \trianglelefteq G$ and $H \leq G$ then $N \cap H \leq H$.
But $M \leq N$, $N \trianglelefteq G \neq M \trianglelefteq G$. \red{not transitive, use $D_8$}
(ii) $N_i \trianglelefteq G \ \forall i \in I \Rightarrow \cap_{i \in I} N_i \trianglelefteq G$.

Proposition 1.16 (cf. Proposition 1.4)
If $N_1 \leq N_2 \leq \ldots \leq G$ with $N_n \leq G \ \forall n \in \mathbb{N}$ then $\cup_{n=1}^\infty N_n \trianglelefteq G$.

Let $X \subseteq G$. 
1 INTRODUCTION

Definition 1.17 (cf. Definition 1.6)
The normal closure $\langle \langle X \rangle \rangle$ or $\langle \langle X \rangle \rangle_G$ to be clear, of $X$ in $G$ is $\cap N$ over all $N$ with $X \subseteq N \trianglelefteq G$ and is the smallest normal subgroup of $G$ containing $X$.

Note: We have $X \subseteq \langle X \rangle \leq \langle \langle X \rangle \rangle$ but $\langle \langle X \rangle \rangle$ can be much bigger than $\langle X \rangle$.

Proposition 1.18 (cf. Proposition 1.9)
If $X \subseteq G$ then the elements of $\langle \langle X \rangle \rangle$ are (e and)

$$g_1 x_1^{n_1} g_1^{-1} g_2 x_2^{n_2} g_2^{-1} \cdots g_k x_k^{n_k} g_k^{-1}$$

for $n_1, \ldots, n_k \in \mathbb{Z}$, $x_1, \ldots, x_k \in X$ and $g_1, \ldots, g_k \in G$ (but not necessarily distinct).

SET PRODUCTS: If $A, B \leq G$ then the set product $AB = \{ab : a \in A, b \in B\}$ is not in general a subgroup of $G$.

Proposition 1.19
(i) $AB \leq G \iff AB = BA$.
(ii) If so then $AB = \langle A, B \rangle$.
(iii) For $N \trianglelefteq G$ and $H \leq G$ we have $NH = HN$.

HOMOMORPHISMS: A function $\theta : G \rightarrow H$ for groups $G, H$ is a homomorphism if $\theta(xy) = \theta(x)\theta(y)$ $\forall x, y \in G$. It's an isomorphism if $\theta$ is bijective (both surjective=onto and injective=1-1) which is equivalent to $\exists$ an inverse $\theta^{-1} : H \rightarrow G$ (inverse here means two sided inverse), in which case $\theta^{-1}$ is unique and is also a homomorphism. We write $G \cong H$, they are then the same as abstract groups.

SETS AND FUNCTIONS: If $X, Y$ are sets and $f : X \rightarrow Y$ is a function then for $U \subseteq X$ the image (or pushforward) of $U$ is $f(U) = \{f(x) : x \in U\} \subseteq Y$, and for $V \subseteq Y$ the inverse image (or pullback) of $V$ is $f^{-1}(V) = \{x \in X : f(x) \in V\} \subseteq X$.

Lemma 1.20
(i) $f^{-1}(f(U)) \supseteq U$ and is equal if $f$ is injective.
(ii) $f f^{-1}(V) \subseteq V$ and is equal if $f$ is surjective.

Theorem 1.21 If $\theta : G \rightarrow H$ is a homomorphism and $A \leq G$, $B \leq H$, we have $\theta(A) \leq H$, $\theta^{-1}(B) \leq G$ and if $B \leq H$ then $\theta^{-1}(B) \trianglelefteq G$. If $\theta$ is surjective then $A \trianglelefteq G \Rightarrow \theta(A) \leq H$. 
Consequently the image $\theta(G)$ of $\theta$ is a subgroup of $H$ and the kernel $\ker \theta = \theta^{-1}(I)$ is a normal subgroup of $G$. In fact we have

Corollary 1.22 For $\theta : G \to H$ with $A \leq G$, $B \leq H$ and $K = \ker \theta$ we have $\theta^{-1}(A) = AK$ and $\theta^{-1}(B) = B \cap \theta(G)$.

QUOTIENTS AND THE ISOMORPHISM THEOREMS: If $N \trianglelefteq G$ then the set of (left) cosets forms a group under the well defined multiplication $xN \cdot yN = (xy)N$. We say the group $G/N$ is a quotient of $G$.

Theorem 1.23 (Homomorphism Theorem)
If $\theta : G \to H$ is a homomorphism then $G/\ker \theta \cong \theta(G)$.

What are the subgroups of $G/N$? If $N \leq H \leq G$ then $N \leq H$ and $H/N \leq G/N$.

Theorem 1.24 (Correspondence Theorem)
If $N \trianglelefteq G$ then the subgroups of $G/N$ are exactly $H/N$ for $N \leq H \leq G$ and the normal subgroups are exactly $L/N$ for $N \leq L \leq G$.

Note: If $\pi : G \to G/N$ is the natural homomorphism and $H \leq G$ then $\pi(H) = HN/N$.

Theorem 1.25 (Product Isomorphism Theorem)
If $N \leq L \leq G$ and $L \trianglelefteq L$ then $H/(H \cap N) \cong HN/N$.

Theorem 1.26 (Quotient Isomorphism Theorem)
Let $N, L \leq G$ with $N \leq L$. Then $(G/N)/(L/N) \cong G/L$.

Definition 1.27 Given $\theta : G \to H$ and $N \trianglelefteq G$ with $\pi : G \to G/N$ the natural homomorphism, we say that $\theta$ factors through $G/N$ if $\exists \bar{\theta} : G/N \to H$ with $\theta = \bar{\theta} \pi$. We must have $N \leq \ker \theta$ and this is sufficient by setting $\bar{\theta}(gN) = \theta(g)$.

EXTENSIONS: If $G/N \cong Q$ we say that $G$ is an extension of $N$ by $Q$. We write $G$ is $N$-by-$Q$ but this does not necessarily determine $G$ uniquely!

Lemma 1.28 If $G$ is $A$-by-$(B$-by-$C)$ then $G$ is $(A$-by-$B$)-by-$C$.


\[ A_4 \cong Q, \quad B \cong C \]
\[ \text{Thus } B = N/A \text{ for } N \trianglelefteq G \text{ (correspondence)} \]
\[ \text{So } C = N/A \cong N/NA \]

Converse: $A_4 = (C_2 \times C_2) \times C_3$
GROUP PROPERTIES: These only depend on the abstract structure of the group. It is always worth asking whether a group property is preserved by (i) Subgroups, (ii) Quotients, (iii) Extensions (namely if \( G/N \cong Q \) and \( N \) and \( Q \) both have this property then \( G \) does too).

For instance, what about finite, countable, cyclic, abelian, finitely generated, max?

**Theorem 1.29** The properties finitely generated and max are preserved by extensions.

**ACTIONS:** We say the group \( G \) acts on the set \( X \) (on the left) if there is a function \( \psi : G \times X \rightarrow X \) such that

\[
\psi(e, x) = x \text{ and } \psi(g_1, \psi(g_2, x)) = \psi(g_1 g_2, x) \forall g_1, g_2 \in G, \forall x \in X.
\]

Note for each \( g \in G \) the function \( x \mapsto \psi(g, x) \) is a permutation of \( X \) (put in \( g^{-1} \), \( g \) and then \( g, g^{-1} \) to get an inverse).

Equivalently there is a homomorphism \( \rho : G \rightarrow S(X) \) given by \( \rho(g)(x) = \psi(g, x) \).

We say \( G \) acts **faithfully** (effectively) if \( \rho \) is injective. We can then unambiguously write \( g(x) \) for \( \rho(g)(x) \); we sometimes do this anyway. We say the action is **free** if \( \rho(g)(x) = x \Rightarrow g = e \) (which implies faithful).

For \( x \in X \) the **orbit** \( \text{Orb}(x) = \{ \rho(g)(x) : g \in G \} \subseteq X \) and the **stabiliser** \( G_x = \{ g \in G : \rho(g)(x) = x \} \subseteq G \). The orbits form a partition of \( X \) and the action is **transitive** if there's one orbit. If \( y = \rho(g)(x) \) then \( G_y = g G_x g^{-1} \) so stabilisers are rarely normal.

**Theorem 1.30 (Orbit-Stabiliser)**

If \( G \) acts on \( X \) then for \( x \in X \) the sets \( \text{Orb}(x) \) and \{ left cosets of \( G_x \) \} are in bijection.

**Example 1.31**

(i) \( G \) acts on itself via \( \rho(g)(x) = gx \). This is transitive, and also free so \( G \leq S(G) \).

(ii) \( G \) acts on itself by conjugation: \( \rho(g)(x) = gxg^{-1} \). Here \( \text{Orb}(x) \) is the **conjugacy class of \( x \)** while the stabiliser is the **centraliser** \( \{ g \in G : gx = xg \} \) of \( x \) in \( G \), written \( C_G(x) \) or sometimes just \( C(x) \) when clear. Note \( \langle x \rangle \leq C(x) \) but it's not generally abelian.

---

i) **Finitely generated preserved by extension**

**Proof**

If \( GN = G, N = \langle n_1, \ldots, n_r \rangle, Q = \langle g_1, N, \ldots, g_s N \rangle \), then \( gN = g_0 N \) for \( g_0 \in \langle g_1, \ldots, g_s \rangle \) so \( g = g_0 n \).

---

ii) **Max is preserved by extension**

**Proof**

Any subgroup of \( G \)

\[
\frac{HN}{HN} \cong \frac{HN}{N} \leq G \text{ max}
\]

Use i)
We also have for $H \leq G$ the centraliser $C_G(H) = \{g \in G : gh = hg \ \forall h \in H\} = \cap_{h \in H} C_G(h)$. We set $C_G(G) = Z(G)$, the centre of $G$, which is abelian and normal.

Moreover $G$ acts on the set of its subgroups by conjugation: $\rho(g)(H) = gHg^{-1} \leq G$. Then $\text{Orb}(H)$ is the set of subgroups conjugate to $H$ and the stabiliser is the normaliser $N(H) = \{g \in G : gHg^{-1} = H\} \leq G$. It is the largest subgroup of $G$ in which $H$ is normal. Also $C(H) \leq N(H)$.

**AUTOMORPHISMS:** An isomorphism (homomorphism) from $G$ to $G$ is an automorphism (endomorphism).

**Example 1.32** For any $g \in G$, $\alpha_g(x) = gxg^{-1}$ is an automorphism so $H \cong gHg^{-1}$. These are the **inner** automorphisms and form a group $\text{Inn}(G)$ under composition.

We have $\alpha_e = e \Leftrightarrow g \in Z(G)$ so $G/Z(G) \cong \text{Inn}(G)$. Moreover all automorphisms form a group $\text{Aut}(G)$, with $\text{Inn}(G) \leq \text{Aut}(G)$ and the quotient is defined to be the **outer** automorphism group $\text{Out}(G)$ of $G$.

**Definition 1.33** The subgroup $C$ of $G$ is characteristic in $G$ if $\alpha(C) = C \ \forall \alpha \in \text{Aut}(G)$ (but $\alpha(C) \leq C \ \forall \alpha$ is enough), so $C \leq G$.

**Proposition 1.34**
(i) A characteristic in $B$, $B$ characteristic in $C$ $\Rightarrow$ A characteristic in $C$.
(ii) A characteristic in $B$ and $B \triangleleft C$ $\Rightarrow$ $A \triangleleft C$.

**DIRECT PRODUCTS:** We can form the direct product $G_1 \times G_2$ from groups $G_1, G_2$ via $(x_1, x_2) \cdot (y_1, y_2) = (x_1y_1, x_2y_2)$. This is external, we can also do this internally.

**Proposition 1.35** If $M, N \leq G$ with $MN = G$ and $M \cap N = I$ then $\theta : M \times N \rightarrow G$ given by $\theta(m, n) = mn$ is an isomorphism.

We can extend this drastically:

**Definition 1.36** Given groups $G_n$ for $n \in \mathbb{N}$, the Cartesian (unrestricted) product is the set of all sequences with $n$th component an element of $G_n$ and pointwise multiplication; we write $\prod_{n \in \mathbb{N}} G_n$. The direct product $\times_{n \in \mathbb{N}} G_n$ is the subgroup of sequences which are eventually $e$. 
Note: If \( G_n \neq I \) for infinitely many \( n \) then \( \times G_n \) is infinitely generated and \( \prod G_n \) is uncountable.

**SEMINDIRECT PRODUCTS:**

**Definition 1.37** Given groups \( G_1, G_2 \) and a homomorphism \( \phi : G_2 \to Aut(G_1) \) then the semidirect product \( G_1 \rtimes_\phi G_2 \) is the set of ordered pairs with multiplication

\[
(x_1, x_2) \cdot (y_1, y_2) = (x_1(\phi(x_2)(y_1)), x_2y_2).
\]

**Example 1.38**

(i) \( \phi \) the trivial homomorphism gives the direct product.

(ii) Take \( \mathbb{Z} = \langle z \rangle \) (written additively) and \( C_2 = \{e, c\} \). Then \( \mathbb{Z} \rtimes_\phi C_2 \) with \( \phi(c)(z) = -z \) is the infinite dihedral group.

**Proposition 1.39** (cf. Proposition 1.35)

If \( H \leq G \) and \( N \trianglelefteq G \) with \( NH = G \) and \( N \cap H = I \) (so \( G/N \cong H \) by 1.25) then \( \theta : N \rtimes_\phi H \to G \) given by \( \theta(n, h) = nh \) and \( \phi(h)(n) = hnh^{-1} \in N \) is an isomorphism.

So again the internal and external versions are equivalent. There is another point of view:

If \( G/N = Q \) with \( \pi : G \to G/N \) the natural projection, we say that the extension **splits** if \( \exists H \leq G \) such that \( \pi : H \to Q \) is an isomorphism.

Now for \( G = NH \) a semidirect product, \( \pi \) restricted to \( H \) is an isomorphism as \( H \cap \ker \pi = I \). But a split extension implies \( H \cap N = I \) and \( NH = G \) as \( \pi(H) = \pi(G) \), so they're the same.

**Example 1.40**

\( SL(2, \mathbb{C}) = \{ A \in GL(2, \mathbb{C}) : \det A = 1 \} \leq GL(2, \mathbb{C}) \) with quotient \( \mathbb{C} - \{0\} \). If \( I_2 \) is the \( 2 \times 2 \) identity matrix then \( PSL(2, \mathbb{C}) = SL(2, \mathbb{C})/\{ \pm I_2 \} \).

This extension does not split because we have elements \( g \) (of order four) in \( SL(2, \mathbb{C}) \) with \( \pi(g) \) of order two in \( PSL(2, \mathbb{C}) \), but the only element of order 2 in \( SL(2, \mathbb{C}) \) is \(-I_2\) with \( \pi(-I_2) = e \).

\[
\text{order } 2 \not\supset \text{det} 1, \ A=A^{-1}
\]

**Example 1.41**

Let \( H = S_0(\mathbb{Z}) \leq S(\mathbb{Z}) \) not \( A_4 \). But \( H \) is generated by \( \{ f(n, n+1) : n \in \mathbb{Z} \} \) as any \( h \in H \) is a product of transpositions.

Now consider \( G = \langle H, A^g \rangle, \ f(z) = z+1 \)

Then \( (n, n+1) = f^n(0, 1)^{-1}, \infty \)

\( G = \langle (0, 1), f \rangle \)

\( G \) has 2 generators, \( H \leq G \), \( H \not\supset \langle f \rangle \).

\[
\text{order } 2 \not\supset \text{det} 1, \ A=A^{-1}
\]
A Brief Guide to Abelian Groups

F.g. abelian groups.

Theorem 2.1 (Structure theorem for f.g. abelian groups)

G f.g. abelian. Then:

(Rational) \( G \cong \mathbb{Z}^r \times C_{d_1} \times \ldots \times C_{d_s} \) with \( 1 < d_1, d_2, \ldots, d_s \) (uniquely)

(PriMary) \( G \cong \mathbb{Z}^r \times P_1 \times \ldots \times P_t \)

where for each \( P_i \) we have a different prime \( p_i \) such that

\[ P_i = C_{p_i^{e_i}} \times \ldots \times C_{p_i^{e_s}} \text{ (uniquely), } 1 \leq e_1 \leq \ldots \leq e_s \]

Corollary 2.2

G f.g. abelian \( \implies \) G has max.

Proof

\( \mathbb{Z} \) has max, so \( \mathbb{Z}^k \) has max (extensions).

Now \( \mathbb{Z}^r \times S \to G \) in 2:1 \( \implies \) G has max. \( \square \)
Let $G$ be a group. $G$ is torsion if all elements have finite order. $G$ is torsion-free if only $e$ has finite order.

If $G$ is f.g., let $d(G) = \min$ size of a generating set.

**Proposition 2.3**

For f.g. abelian $G = \mathbb{Z}^r \times C_{d_1} \times \ldots \times C_{d_s}$ as in 2.1, $d(G) = r + s$.

**Proof**

We have $d(G) \leq r + s$. Take some prime $p | d_1$, then

$\exists \theta : G \rightarrow (C_p)^{r+s}$. $\theta(G)$ is a vector space over $\mathbb{F}_p$, and a generating set here is the same as a spanning set.

$\exists (\text{gen. set for } G) = \text{gen. set for } \theta(G).$

Non f.g. Abelian Groups

E.g. $\mathbb{Q}$, $\mathbb{Q}/\mathbb{Z}$, $\mathbb{R}$, $\mathbb{R}/\mathbb{Z}$

If $p$ is a property preserved by subgroups, we say that $G$ is locally $p$ if $H$ has $p$ for all f.g. $H \leq G$.

**Examples**

1. $H = \left\langle \frac{p_1}{q_1}, \ldots, \frac{p_k}{q_k} \right\rangle \leq \mathbb{Q} \Rightarrow H \leq \left\langle \frac{1}{q_1 \cdots q_k} \right\rangle$

   \therefore $\mathbb{Q}$ is locally cyclic (torsion-free)

2. $\mathbb{Q}/\mathbb{Z}$ is locally cyclic, and $p\mathbb{Q} + \mathbb{Z}$ has finite order, so $\mathbb{Q}/\mathbb{Z}$ is locally finite (torsion).
Inve. I

If $A \times Z \cong B \times Z$ then we can have $A \neq B$, but if this is abelian, then in fact $A \cong B$.

Proofs (of 2nd part)

Suppose that $I' = A \times Y = B \times Z$, $Y, Z \cong Z$.

i) If $A \leq B$, then $\frac{I'}{A} \cong Z \to \frac{I'}{B} \cong Z$

This induces an isomorphism. If $b \in B \setminus A$, $b$ is in the kernel, but we know that the kernel must be trivial. Hence $A = B$.

ii) If $A \neq B$, $B \neq A$, then

$$\frac{A}{A \cap B} \cong \frac{AB}{B} \leq \frac{I'}{B} \cong Z,$$

and $B \leq AB$.

So $A \cong (A \cap B) \times Z$, $B \cong (A \cap B) \times Z$ similarly.

Hence $A \cong B$.

Abelianisation

Let $G$ be a group. Let $x, y \in G$.

The commutator $[x, y] := xyx^{-1}y^{-1} \in G$.

Definition 2.4

The commutator/derived subgroup $G'$ of $G$ is

$$\langle \{[x, y] : x, y \in G \} \rangle$$

Proposition 2.5

$G' \leq G$ with $G / G'$ abelian.

Proof

For $\alpha$, an automorphism of $G$, $\alpha([x, y]) = [\alpha(x), \alpha(y)] \in G'$

So $G'$ is characteristic in $G$, and characteristic $\Rightarrow$ normal.
In $G'$, $xyG' = yx [x^{-1}, y^{-1}]G'$.  

**Corollary 2.6**

$G_{/G'}$ is the largest abelian quotient of $G$. If $G_{/N}$ is abelian then $G_{/N} \rightarrow G_{/G'}$ factors through $G_{/N}$.  

**Definition 2.7**

For any $G$, the abelianization of $G$ is this abelian quotient $G_{/G'}$.  

3 Free Groups  

**Definition 3.1**

Let $X = \{a, b, \ldots \}$ be a set of symbols. Let $X^\pm = \{a^+, b^+, \ldots \}$ be a set disjoint from $X$ and in bijection with $X$. We write $X^\pm = X \cup X^-$. A word in $X^\pm$ is a finite sequence of elements of $X^\pm$ (letters) including $\varnothing$. Let $W$ be the set of all words. Let $W_0 \subseteq W$ be the set of reduced words i.e. words containing no subword $a^+ a^-$ or $x^{-1}x$.

We want to define the free group on $X$ as the set of reduced words with multiplication $W_0 \cdot W_0 = W, W_2$, but proofs are difficult this way.  

**Definition 3.2**

The free group $F(X)$ on set $X = \{x_i : i \in I\}$ is a subgroup of $S(W_0)$ generated by elements $x_i$.  

symmetric group on reduced words
\[ X_i(w) = \begin{cases} x_i w & \text{if } w \text{ does not start with } x_i^- \\ w & \text{if } w = x_i^- w \end{cases} \]

\[ X_i^{-1}(w) = \begin{cases} x_i^- w & \text{if } w \text{ does not start with } x_i \\ w & \text{if } w = x_i w \end{cases} \]

**Proposition 3.3**

The map \( M: W_o \to F(X) \) given by replacing \( x_i^+ \) by \( x_i^- \) and multiplying in \( F(X) \) is an injection.

**Proof**

If \( M(w_1) = M(w_2) \) for \( w_1, w_2 \in W_o \) then note that \( M(w_i)(\emptyset) = w_i \) by (3.2)

**Corollary 3.4**

\( M: W_o \to F(X) \) is surjective, and given a word \( w \in W_o \), if we delete all subwords \( xx x^- \), \( x^- x \) in any order, we reach a unique word \( w_0 \in W_o \).

**Proof**

Extend \( M: W \to F(X) \), surjective by (1.9). Each deletion reduces the length of \( w \), so we reach some \( w_0 \in W_o \).

Deletions do not change the group element, so \( M(w) = M(w_0) \), so \( w_0 \) is unique by (3.3).

**Word \rightarrow Image in \( F(X) \) under \( M \)**

**Unique reduced preimage**

\( \sim M: W_o \to F(X) \) surjective
Theorem 3.5
(Universal Property of free groups)
Any \( f : X \rightarrow G \), a group, extends uniquely to a homomorphism \( f^* : F(X) \rightarrow G \), so that \( f^*(x_i) = f(x_i) \).

Proofs
First, define \( f^*(x_i^{-1}) = f(x_i)^{-1} \in G \) \( \{ \text{(1)} \} \)
then let \( f^*(t_1 \ldots t_k) = f(t_1) \ldots f(t_k) \) \( \{ \text{(x) \}} \)
where \( t_j \in \{ X, X^{-1} \} \) is represented by letter \( L_i \in X^{+1} \).
Then \( f^* \) is well-defined by (3.3), (3.4) and is a homomorphism by definition. Clearly, any such homomorphism must satisfy \( \{ \text{(x)} \} \)
\[ f : x \rightarrow y \]
\[ g : y \rightarrow x \]
\[ f \circ g = id_y \]
\[ g \circ f = id_x \]

**\( f \) injective:**
\[ f(x_1) = f(x_2) \]
\[ \Rightarrow g(f(x_1)) = g(f(x_2)) \]
\[ \Rightarrow x_1 = x_2 \]

**\( f \) surjective:**
\[ y \in Y \]
\[ g(y) \in X \]
\[ f(g(y)) = y \]

\[ \therefore f \text{ bijective (similarly for } g) \]
Proposition 3.6

If \( F(x) \) and \( F(y) \) are free groups on \( x, y \), then
\[ F(x) \cong F(y) \iff |x| = |y| \]

Proof

\( (\Leftarrow) \) If \( f : x \to y \) is a bijection then we can extend to a homomorphism
\[ f^* : F(x) \to F(y) \] and \( (f^{-1})^* : F(y) \to F(x) \).

But \( (f^{-1})^* f^* \) fixes \( x \) so this is the identity homomorphism and
the same for \( f^*(f^{-1})^* \) so bijective.

\( (\Rightarrow) \) For any group \( G \), let \( S_G < G \) be \( \langle g^2 : g \in G \rangle \).

Then \( S_G \) is abelian with all non-identity elements of order 2, this
is a vector space over \( \mathbb{F}_2 \). Now the image of \( x \) in \( F(x) \)
\[ \text{linearly independent and spans } |x| \text{ is the dimension.} \]

We can define \( F_n \) (the free group of rank \( n \)) when \( |x| = n \):
\[ F_0 = 1, \quad F_1 = \mathbb{Z}, \quad \text{but if } a, b \in x, \quad a \neq b, \quad \text{then} \]
\[ ab \neq ba \text{ in } F(x) \] so these are non-abelian.

In fact \( \langle a, b \rangle = F_2 \leq F(x) \).

Corollary 3.7

Every (finitely generated) group is a quotient of a
(finitely generated) free group.

Proof

If \( G = \langle g_i : i \in I \rangle \) then take \( X = \{ x_i : i \in I \} \) and
use 3.5. \( f : X \to G \)
\[ x_i \mapsto g_i \] extends to a homomorphism \( f^* \). \( \square \)
Corollary 3.8

If \( \Theta : F_n \rightarrow F_n \) then \( F_n \leq G \) and \( C = \ker \Theta \times F_n \).

Proof

Take \( g_1, \ldots, g_n \in G \) with \( \Theta(g_i) = x_i \); then extend \( f(x_i) = g_i \). We have \( \Theta f = \text{id} \) is the identity homomorphism (unique).

so \( f : F_n \rightarrow G \) is an isomorphism from \( F_n \) to \( f(F_n) \).

For \( K = \ker \Theta \), \( Kf(F_n) = G \) and \( Kn f(F_n) = 1 \) \((139)\). \( \square \)

Definition 3.9

We say that the set \( S = \{ s_i \} \subseteq F(x) \) is a free basis for \( F(x) \) if \( \langle S \rangle = F(x) \) (generates) \(" \text{span} \) 

and for any reduced word \( w_0 \neq \emptyset \) on \( S^* \)

we have \( w_0 \neq \text{id} \) when evaluated in \( F(x) \). \(" \text{free} \) \(" \text{linearly independent} \) \( \square \).

Proposition 3.10

Free bases for \( F(x) \) have cardinality \( 1 \times 1 \).

Proof

Given a free basis \( S \) for \( F(x) \), define a homomorphism \( f^* : F(S) \rightarrow F(x) \) by extending \( s \in S \mapsto s \in F(x) \).

Then \( f^* \) is injective (generates) and injective (free).

Then by 3.6, \( F(x) \cong F(S) \Rightarrow |S| = 1 \times 1 \) \( \square \).

Proposition 3.11

The automorphisms of \( F(x) \) are exactly (extension of) bijective functions \( f : x \rightarrow S, \) \( S \) free basis.
Proof

An automorphism \( \alpha \) must send a free basis bijectively to a free basis as \( \alpha(w(x_i)) = w(\alpha(x_i)) \) and it is determined by this. Moreover, given \( f \), extend to \( f^* : F(X) \rightarrow F(X) \) uniquely. Then \( f(x) = s \) means that \( f^* \) is injective and \( s \) free says that \( f^* \) is 1-1. So every free basis mapping gives an automorphism.

**Definition 3.12**

A word in \( x^+ \) is cyclically reduced if it is reduced and the first and last letters of \( w \) are not inverses.

We can write any reduced word \( w_0 = w_1 w_1^{-1} w \) where \( w_1 \) is cyclically reduced and \( 1 \) means no cancellation between 2 words.

**Proposition 3.13**

If \( w, w' \) are cyclically reduced then they are conjugate

\( \Leftrightarrow w' \) is a cyclic permutation of \( w = l_1 l_2 \cdots l_n \) for \( li \in x \).

**Proof**

(i) (ii)

If \( w' = c wc^{-1} \) (\( c \) reduced, \( c \neq 0 \)) then as \( w' \) is cyclically reduced, we have some cancellation at (i), or (ii), but not both (as \( w \) is cyclically reduced). WLOG, say \( c = d l_1^{-1} \) \( w = l_1 l_2 \cdots l_n \), then \( w' = d l_1 d^{-1} l_1 \) but \( l_1 \) is a cyclic permutation of \( w \) so either \( d = 1 \) or we continue the process. N.B. Continue (N.B. terminate as length \( c \) decreases each time)

\( w = l_1 l_2 \cdots l_n \)

Converse:

\( w' = l_1 l_2 \cdots l_n l_1 l_2 \cdots l_n l_{n-1} \)

\( w' = (l_{n-1} \cdots l_1) w (l_1 \cdots l_{n-1}) \)
Example 3.15
Take \( F_2 \) free on \( a, b \). Let \( H_n = \langle a, ba^{-1}, ..., b^{-1}a^{-1} \rangle \leq F_2 \).

Corollary 3.14
A free group is torsion-free.

Proof
Any reduced word \( w_0 (\neq \emptyset) = u_1 w_1 u_1^{-1} \) where \( w \) is cyclically reduced. But for \( n > 0 \), \( w_0^n = u_1 w_1 u_1^{-1} \) so we have no cancellation, so \( w_0^n \neq \emptyset \).

Example 3.15 (continued)
By 1.9, if \( H_n = H \), then \( H = b^{-i}a^{-k}b^{-i-1}a^{-k}b^{-i-1} ... b^{-i-m} \)
for \( i < i_j \leq n \). Hence \( b^{-i}a^{-k}b^{-i-1} ... b^{-i-m} \leq H_n \).

So \( H = \bigcup H_n \) is not finitely generated by 1.11, because it does not have max.

Corollary 3.16
A finitely generated group \( G \) containing a non-abelian free group \( (\Rightarrow F_2 \leq G) \). So \( \emptyset \) does not have max.

Free Products
Definition 3.17
Let \( \{ G_a : a \in A \} \) be an indexed family of groups.
A reduced sequence in \( \{ G_a \} \) is a finite sequence \( g_1, ..., g_n \)
where \( g_i \in \bigsqcup_{a \in A} G_a \). Each \( g_i \neq e \) and no successive \( g_i, g_{i+1} \) are in the same \( G_a \).

Let \( \mathcal{R} \) be the set of all reduced sequences (including \( \emptyset \)).
Definition 3.18

The free product $\ast_{\lambda \in A} G_{\lambda}$ is the subgroup of $S(\mathbb{R})$ generated by elements $r(g, \lambda)$ for $\lambda \in A$ and $(g, \lambda) \in G_{\lambda} \setminus I$, where

$$r(g, \lambda)(g_1, \ldots, g_n) = \begin{cases} f(g, \lambda)g_1, \ldots, g_n & \text{if } g \neq e_{G_{\lambda}} \\ (g, \lambda)g_1, \ldots, g_n & \text{if } g = e_{G_{\lambda}} \end{cases}$$

with $r'(g, \lambda) = r(g', \lambda)$

Proposition 3.19

The function $f : \mathbb{R} \rightarrow \ast_{\lambda \in A} G_{\lambda}$ given by

$$f(g_1, \ldots, g_n) = rg_1 \circ \ldots \circ rg_n$$

is a bijection.

Proof

If $g_1, g_2 \in G_{\lambda}$ then $rg_1 \circ rg_2 = rg_1g_2$ as gather Rs from some group to get a bijection (remove e).

Then $f(g_1, \ldots, g_n)(\emptyset) = rg_1 \circ \ldots \circ rg_n(\emptyset) = g_1 \ldots g_n \quad \square$

Proposition 3.19

The function $f : A \rightarrow \ast_{\lambda \in A} G_{\lambda}$

$$f(g_1, \ldots, g_n) = rg_1 \circ \ldots \circ rg_n$$

is a bijection on $A$.

Proof

If $g_1, g_2 \in G_{\lambda}$, the same group, then $rg_1 \circ rg_2 = rg_1g_2$.

Collect a large enough set of Rs so that the map is injective.

$f(g_1, \ldots, g_n)(\emptyset) = rg_1 \circ \ldots \circ rg_n(\emptyset) = g_1 \ldots g_n$

so this map is injective.
3.17) The set of all sequences in disjoint union.

3.19) \( f : A \to \star G \) given \( f(\ldots) = \nu \) \((\nu, e, (\nu \cdot e, \lambda) = \text{Id}_{\mathcal{S}(R)}\)

restrict to a bijection from \( R \).

Note

\( \lambda : G \to \star G \) given by \( \lambda(g) = (g, \lambda) \) is an injective
homomorphism.

Theorem 3.20 (Universal Property)

For any group \( H \) and any collection of homomorphisms

\( \Theta : G \to H \), \( \exists ! \Theta : \star G \to H \), s.t. \( \Theta \circ \lambda \Theta \).

Proof

Define \( \Theta(g_1 \ldots g_n) = \Theta_1(g_1) \ldots \Theta_n(g_n) \) where \( g_i \in G_i \).

Similarly, this is a homomorphism and is unique. \( \square \)

If \( F(x) \) is free on \( X = \{ x_i : i \in I \} \) then it is \( \star G \)

where \( G_i = \{ x_i^n : n \in \mathbb{Z} \} \) infinite cyclic.

Note that \( G_1 \times G_2 \) is infinite if \( G_1, G_2 \neq I \), as

\( (g_1, g_2)^n \neq e \) for \( n > 0 \) and \( g_2 g_1 \neq g_1 g_2 \).

Suppose that \( X \) is a topological and \( G \) the group of homeomorphisms of \( X \). We say that \( S \subseteq X \) is a \( G \)- packing if \( g(S) \cap S = \emptyset \)

\( \forall g \in G \).

Theorem 3.21 (Klein, 1883)

If \( G_1, G_2 \leq \text{Homeo} (X) \) with \( G_i \) packings \( S_i \) s.t. \( S_1 \cup S_2 = X \) and \( S_1 \cap S_2 \neq \emptyset \) then \( G = \langle G_1, G_2 \rangle = G_1 \star G_2 \).
Proof

Note for \( x \in S_1 \), \( g(x) \not\in S_1 \) for any \( g \in G \). \( \forall x \in g(x) \not\in S_2 \).

Take \( S \in S_1 \cap S_2 \) and a reduced sequence \( g_1, \ldots, g_n \) with (WLOG) \( g_n \in G \).

Then \( g_n(s) \in S_2 \setminus S_1 \), \( g_{n-1}g_n(s) \in S_1 \setminus S_2 \), and so on, so that \( g_1 \ldots g_n(s) \not\in S \). Thus, the homomorphism in (3.20)

\[ \theta : G_1 \times G_2 \rightarrow \langle G, G_2 \rangle \]

is an isomorphism. □

the above ensures injectivity as \( \theta \) has trivial kernel.

Examples

1. Let \( X = \mathbb{R}^2 \) with reflections \( a, b \) in lines \( \alpha, \beta \).

Then \( \langle a, b \rangle = \mathbb{Z}_2 \times \mathbb{Z}_2 \).

This is an infinite dihedral group (1.38) as \( ba(x) = x+1 \)

and \( a^{-1} (ba)a^{-1} = (ba)^{-1} \).

2. Möbius transformations, \( f(z) = \frac{az+b}{cz+d} \), \( a, b, c, d \in \mathbb{C} \), \( ad-bc \neq 0 \)

is a bijection of \( \mathbb{C} \cup \{ \infty \} \) with inverse \( f^{-1}(z) = \frac{dz+b}{-cz+a} \), so they form a group.

\[ f(z) = z+2, g(z) = \frac{z}{z+1} \] both preserve the (open) upper half plane \( U \).

Proposition 3.23

\[ F = \langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, G = \langle \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \rangle \in SL_2(\mathbb{Z}) \cap SL_2(\mathbb{Z}) \]

\[ \cong \langle F, G \rangle \]

Proof

\( f, g \in PSL_2(\mathbb{Z}) \) by (3.21), and

\( \langle F \rangle, \langle G \rangle \) are infinite, so \( \langle F, G \rangle = F_2 \).
Now send $f, g \mapsto F, G$ and extend (3.5) to $\Theta$ with $\Theta \circ \Theta = \text{id}$, so $\Theta(F) = F$.

**Definition 3.24**

A graph $\Gamma'$ (a 1-d CW complex) is made up of a set $V$ of vertices with discrete topology and a set $E$ of edges $I_a$ (with interior $I_a = E_a$). Each $I_a$ is a copy of $[0, 1]$. The edge endpoints are attached to points in $V$, giving $f_0, f_1 : E \to V$ so that $\Gamma = \bigcup_{a} E_a / I_a$. 

1. $S \subseteq \Gamma'$ is open (closed) $\iff$ $S \cap E_a$ open (closed) $\forall a$.
2. A basic open neighborhood of $V \subseteq \Gamma'$ is $\Gamma$ is locally path connected, locally contractible.
   (connected $\iff$ path connected)
3. Basic open sets for $\Gamma$ are these + open intervals in $E_a$. 

Not allowed this type!
Topics in Infinite Groups
Topological Background

We follow the books Lee and Hatcher as in the course summary.

$X$ is a topological space which is always assumed to be path connected and \textbf{locally path connected}: \[ \forall x \in X \text{ and } \forall \text{ open } U \subseteq X \text{ with } x \in U, \text{ we have an open path connected set } P \text{ with } x \in P \subseteq U. \]

Let $f, g : X \to Y$ be continuous maps (and let $A \subseteq X$).

A \textbf{homotopy} between $f$ and $g$ (relative to $A$, written rel $A$) is a continuous map $H : X \times [0, 1] \to Y$ with $H(., 0) = f$ and $H(., 1) = g$ (with $H(a, t) = f(a)$ \[ \forall a \in A \text{ and } \forall t \in [0, 1], \text{ so } f \text{ and } g \text{ must agree on } A \]). This is an equivalence relation, written $f \simeq g$ (or $f \simeq g$ rel $A$).

A (strong) \textbf{deformation retraction} of $X$ onto $A \subseteq X$ is a homotopy rel $A$ from $Id_X$ to $r : X \to X$ with $r(X) \subseteq A$ and $r|_A = Id_A$.

For $x_0 \in X$ the \textbf{fundamental group} $\pi_1(X, x_0)$ is the group of homotopy classes of closed paths $\gamma$ with start and end $x_0$ (in other words $\gamma : [0, 1] \to X$ is continuous with $\gamma(0) = \gamma(1) = x_0$). Changing the basepoint $x_0$ "doesn’t matter" as we obtain an isomorphic group, written $\pi_1(X)$.

Any continuous map $f : X \to Y$ induces a homomorphism $f_* : \pi_1(X) \to \pi_1(Y)$. If $X$ is homeomorphic to $Y$ (which is also written $X \cong Y$) then $\pi_1(X) \cong \pi_1(Y)$.

$X$ is \textbf{homotopy equivalent} to $Y$ (written $X \simeq Y$) if there exist continuous maps $f : X \to Y$ and $g : Y \to X$ such that $gf \simeq Id_X$ and $fg \simeq Id_Y$. If so then $f_* : \pi_1(X) \to \pi_1(Y)$ is an isomorphism.

$X$ is \textbf{contractible} if $X \simeq \{x\}$, which implies that $X$ is \textbf{simply connected} (meaning that $\pi_1(X)$ is the trivial group).

\textbf{Seifert - Van Kampen Theorem} (Hatcher Theorem 1.20 in the case where the intersections are simply connected): If $X = \cup_\alpha A_\alpha$ where all of the $A_\alpha$ are path connected and open in $X$, with $x_0 \in \cap_\alpha A_\alpha$ and each pairwise
and triple intersection $A_{\alpha_1} \cap A_{\alpha_2}; A_{\alpha_1} \cap A_{\alpha_2} \cap A_{\alpha_3}$ is simply connected then

$$\pi_1(X) \cong \ast_\alpha \pi_1(A_\alpha).$$

If $A \subseteq X$ and $\iota : A \rightarrow X$ is the inclusion map then sadly $\iota_* : \pi_1(A) \rightarrow \pi_1(X)$ might not be injective or surjective (see picture). But in a deformation retraction $X \simeq A$ (by taking $f = r$ and $g$ as inclusion of $A = Y$ in $X$) so $X$ and $A$ have isomorphic fundamental groups.

A continuous map $p : \tilde{X} \rightarrow X$ is a covering map (with $\tilde{X}$ a covering space for $X$) if $\forall x \in X$ there exists an open neighbourhood $V$ of $x$ with $p^{-1}(V)$ a disjoint union of open sets in $\tilde{X}$, each of which is mapped homeomorphically by $p$ onto $V$. As $X$ is connected, the cardinality of $p^{-1}(x)$ (which is non-zero as we assume $\tilde{X}, X \neq \emptyset$) is constant: the degree or number of sheets.

Given a path $\gamma : [0, 1] \rightarrow X$ and a point $\tilde{x}$ above $\gamma(0)$, there exists a unique path $\tilde{\gamma} : [0, 1] \rightarrow \tilde{X}$ lifting $\gamma$ (namely $p \tilde{\gamma} = \gamma$). Moreover $p_* : \pi_1(\tilde{X}) \rightarrow \pi_1(X)$ is injective.

Now assume $X$ is locally contractible.

Classification of Coverings Theorem: For each $H \leq \pi_1(X)$ there exists a cover $p : \tilde{X} \rightarrow X$ with $p_*(\pi_1(\tilde{X})) = H$ (this cover is "unique"). The degree is the index of $H$ in $\pi_1(X)$.

A deck transformation of the cover $p : \tilde{X} \rightarrow X$ is a homeomorphism $\phi$ of $\tilde{X}$ such that $p \phi = p$. They form a group $D$ and for any $x_0 \in X$ the action of $D$ on $p^{-1}(\{x_0\}) \subseteq \tilde{X}$ is transitive if and only if $p_* \pi_1(\tilde{X}, \tilde{x}_0)$ is normal in $\pi_1(X, x_0)$, in which case we say $p$ is a regular cover.

C.f. Galois Theory!
A subgraph of $\Gamma$ is a union $\Delta$ of edges and vertices such that $e_\alpha \in \Delta \Rightarrow \overline{e_\alpha} \in \Delta$.

**Lemma 3.25**

If $\tilde{X}$ covers the (connected) graph $\Gamma$, then $\tilde{X}$ is a graph with vertices and edges the lifts of those in $\Gamma$.

**Proof**

Let $V(\tilde{X}) = \tilde{r}^{-1}(V)$ for $r: \tilde{X} \to \Gamma$, and for edges, take the map from $I_x$ into $\Gamma$ and $\tilde{r}$ above $\tilde{r}_0(I_x)$ to get a unique lift. Thus $\tilde{X}$ is a graph and the topologies agree on basic open sets, so they are the same.

If $S$ is a simple graph, an edge path in $S$ is a finite sequence of vertices such that any 2 consecutive vertices span an edge in $S$. A cycle $V_0, \ldots, V_n$ is a closed edge path ($V_0 = V_n$).

A tree is a connected simple graph with no reduced cycles.

**Proposition 3.26**

Given any connected graph $\Gamma$, any vertex $V_0$ in $\Gamma$, there exists a subgraph $\Delta = \{V_0\}$ such that $\Delta$ contains all of $V(\Gamma)$.

**Proof**

Let $\Gamma_0 = \{V_0\} \subseteq \Gamma_1 \subseteq \Gamma_2$ be a sequence of subgraphs where $\Gamma_{i+1}$ is $\Gamma_i$ with edges $\overline{e_\alpha}$ for all $e_\alpha \subseteq \Gamma \setminus \Gamma_i$. with an endpoint in $\Gamma_i$. i.e. $\Gamma_{i+1} = \Gamma_i + (\text{neighbors of } \Gamma_i)$

Then $U \Gamma_i$ is open (neighborhood of a point in $\Gamma_i$ is in $\Gamma_i$)
and closed in \( \Gamma \) (union of closed edges), so is all of \( \Gamma \).

Next \( \Delta_0 = \Gamma_0 \subseteq \Delta_1 \subseteq \Delta_2 \subseteq \ldots \) where \( \Delta_i \) has the same edges as \( \Gamma_i \) but \( \Delta_{i+1} = \Delta_i \cup \text{one edge from each } v \in \Gamma_i \setminus \Gamma_i \) to \( \Delta_i \).

Now \( \Delta_i \) deformation retracts onto \( \Delta_i \). So set \( \Delta = \bigcup \Delta_i \) (contains \( V(\Gamma) \)). \( \Delta \) deformation retracts to \( \Delta_0 \) by performing homotopy in the interval \( \left[ \frac{1}{2}, 1 \right] \), with

\[
\text{homotopy } \Delta \Rightarrow \Delta \text{ continuous}
\]

Coindley 3.27

A tree \( T \) is contractible.

Proof

Apply 3.26 to get \( \Delta \subseteq T \) contractible. If edge \( e \) is spanned by \( v \neq w \), \( v \notin \Delta \) then take reduced edge paths \( v_0, \ldots , v_k \) and \( w_0, \ldots , w_{k-1} \) in \( \Delta \). Note that \( u \neq w \), \( v \neq x \) or \( e \notin \Delta \). So \( v_0, \ldots , v_k \) is a reduced cycle in \( T \). So \( \Delta = T \), \( \Delta \) contractible.

Theorem 3.28

The fundamental group of a (connected) graph \( \Gamma \) is free.

Proof

Let \( T = \Gamma \) (WLOG simple) be \( \Delta \) (in 3.26) so this is a tree, with \( \text{Le} : \alpha \in A \) the edges in \( \Gamma \setminus T \) and \( \text{Le} : \text{ endpoints of } \alpha \). For each \( \alpha \), take a reduced cycle


Topics in Infinite Groups

Ca from $V_0$, including $W_0 W_1$ so that $C_\alpha = \ldots W_\alpha W_{\alpha'} \ldots$. 

First assume one edge. If $c \leq_\alpha u_i = v_i$ in it then replace with $u_i \ldots v_i$ which still contains $W_\alpha W_{\alpha'}$, otherwise it is a reduced cycle in $T$. So now we have (renamed) $V_0 \ldots V_\Gamma$ with no repeats giving a loop $L \leq S'$, so $\pi_1 L \leq \mathbb{Z}$.

But $\Gamma \leq L$ as the components $K$ of $\overline{\Gamma \setminus L}$ are in $T$ and so are trees.

If $a \neq b$, $a, b \in K \setminus L$, then go from $a$ to $b$ in $K$ then go from $a$ to $b$ in $K$ the back via $L$.

Thus by 3.27 we can deformation retract each $K$ (simultaneously) onto the point in $K \setminus L$, giving $\pi_1 (\Gamma) = \mathbb{Z}$.

For general $\Gamma$, let $M_a$ be a "midpoint" of $e_\alpha$ and set 

$$A_\alpha = (\Gamma \setminus U \setminus a ) \cup M_a$$

Then $A_\alpha$ is open in $\Gamma$ and path connected with

$$A_\alpha \cap A_\beta \cap (v A_\gamma) = \Gamma \setminus U \setminus M_a$$

which deformation retracts onto $T$ so that it is simply connected by 3.27). Also, $A_\alpha$ retracts onto $T \cup e_\alpha$. So $\pi_1 (A_\alpha) \cong \mathbb{Z}$ giving (Seifert - van Kampen)

$$\pi_1 (\Gamma, v) \cong \mathbb{Z}$$

Corollary 3.29

For every free group $F(x)$ $\exists$ a simple graph $\Gamma$ with $\pi_1 \Gamma = F(x)$
Proof

Take a loop \( L_x \) for each \( x \in X \), all join at \( v \), turn it into a triangle and use 3.28

Theorem 3.30 (Nielsen/Schreier)

A subgroup of a free group is free.

Proof

For \( H \leq F(X) \), use 3.29, subgroups \( \leftrightarrow \) coverings

\((\tilde{X} \rightarrow X, \pi_1(\tilde{X}) = H)\), 3.25 and 3.28). \( \pi_1 \) is free.

Theorem 3.31 (Nielsen-Schreier index formula)

If \( H \) has index \( i \) in the free group \( F_n \) then \( H \) has rank \( i(n-1)+1 \).

Proof

For a finite graph \( \tilde{\Gamma} \), define \( X(\tilde{\Gamma}) = |V| - |E| \).

If \( p: \tilde{\Gamma} \rightarrow \Gamma \) has degree \( i \) then \( X(\tilde{\Gamma}) = i X(\Gamma) \) by 3.25.

Take \( \tilde{\Gamma} \) in 3.29 with \( \pi_1 \tilde{\Gamma} = F_n \) and \( X(\Gamma) = 1 - n \).

For a subgroup \( H \) of index \( i \) in \( F_n \) with \( \pi_1(\tilde{\Gamma}) = H \), we have \( X(\tilde{\Gamma}) = i (1 - n) \). Now in 3.28 note \( X(\Delta) = 1 \), so the set of edges \( \{ e_x \} \) for \( \tilde{\Gamma} \backslash \Delta \) number \( i(n-1)+1 \), so this is the rank of \( H \).

Notes

i) \( F_2 \leq F_3 \), and \( F_3 \leq F_2 \).

ii) If \( N \triangleleft F(X) \), it can be shown that if \( N \) has infinite index and \( N \neq 1 \), then \( N \) has infinite rank.
4. Presentations of Groups

By 3.7 we can write any group $G$ as $\frac{F(x)}{\langle R \rangle}$ where (the image of) $X$ is a generating set for $G$.

**Definition 4.1**

A presentation $\langle X \mid R \rangle$ for a group $G$ is a set $X$ and a subset $R$ of $F(x)$ such that $G \cong \frac{F(x)}{\langle R \rangle}$

Elements of $X$ are generators, of $R$ are relations.

**Theorem 4.2 (von Dyck)**

If $G = \langle X \mid R \rangle$ then $Q$ is a quotient of $G$.

$\Rightarrow$ $Q \cong \langle X \mid R \cup S \rangle$ (quotient = more relations)

**Proof**

$\Rightarrow$ Have $F(x) \xrightarrow{\pi} \langle X \mid R \cup S \rangle$ factors through $G$ as $\langle R \rangle \subseteq \langle R \cup S \rangle$ trivial.

$\Rightarrow$ Suppose $Q = G/\Lambda$ for $G = \frac{F(x)}{\langle R \rangle}$ where $M = \langle R \rangle$.

Then $L = \frac{NM}{M}$ for $N \triangleleft F(x)$ so $Q \cong \frac{F(x)}{NM}$

Take any $S$ with $\langle S \rangle = N$.

Then $\langle R \cup S \rangle = \langle N \cup M \rangle = MN$.

So $Q \cong \langle X \mid R \cup S \rangle$.

**Definition 4.3**

$G$ is finitely presented (f.p.) if

$G = \langle x_1, \ldots, x_n \mid r_1, \ldots, r_m \rangle$

**Proposition 4.4 (B. Neumann 1937)**

If $G = \langle x_1, \ldots, x_n \mid r_1, \ldots, r_m \rangle = \langle y_1, \ldots, y_k \mid s_1, \ldots, s_l \rangle$.

Then \( G = \langle y_1, \ldots, y_k \mid S \rangle \) for some \( S \).

**Proof**

We set \( y_i = v_i(\bar{x}_1, \ldots, \bar{x}_n) \), \( x_i = w_i(y_1, \ldots, y_k) \), thinking of \( x_i, y_i \) as elements of \( G \) of generators.

Thus \( y_i = v_i(w_i(y_i), \ldots, w_n(y_i)) \) (i runs over \( 1 \leq i \leq k \)) and \( w_i(y_i), \ldots, w_n(y_i) = e \in G \).

Let \( N = \langle s_1, s_2, \ldots \rangle \triangleleft F\langle \bar{s}_i \rangle \) and form \( \bar{G} = \langle \bar{y}_1, \ldots, \bar{y}_k \mid \bar{y}_i = v_i(\bar{w}_1(y_i), \ldots, \bar{w}_n(\bar{y}_i)) \rangle \).

\( \exists \Theta: G \to F\langle \bar{s}_i \rangle \) such that \( \bar{y}_i \mapsto y_i \) as each relation holds in \( G \), since \( F\langle \bar{s}_i \rangle \to F\langle \bar{s}_i \rangle \) factors through \( \bar{G} \).

Now take \( \Phi: \langle x_1, \ldots, x_n \mid r_1, \ldots, r_m \rangle \to \bar{G} \) by \( \Phi(x_i) = \bar{x}_i = w_i(\bar{y}_1, \ldots, \bar{y}_k) \) \( \Phi \to \bar{G} \to G \).

Then \( \Theta \Phi(x_i) = x_i \), \( \Phi \) onto, so \( \Theta \) is an isomorphism.

Therefore \( N = \langle \text{finite set} \rangle \) because all normal subgroups of \( N \) are finitely normally generated.

But \( N = \bigcup_{i=1}^{\infty} \langle s_1, \ldots, s_i \rangle \) (use max property) \( \square \).

**Proposition 4.5**

\( G = \langle a, b \mid [a^{2n+1}, b^{-1}]^n, n \in \mathbb{N} \rangle \) is not f.p.

**Proof**

In \( A_5 \), let \( \alpha = (1\ 2\ \ldots\ j) \) and \( \beta = (1\ 2\ 3) \).

Then \( \alpha^k \beta \alpha^{-k} \) commutes with \( \beta \) if \( 3 \leq k \leq j-3 \), but not if \( k = j-2 \). Thus in \( A_{2j+3} \) we have \( c_1, \ldots, c_{j-2} = e \) but \( c_j \neq e \).
So \( C_n \cong \langle c_1, \ldots, c_{n-1} \rangle \leq F_2 \), \( a \rightarrow \alpha \), \( b \rightarrow \beta \). Now use 4.4.5

Aside

The paper shows that \( \exists \) uncountably many f.p. groups up to isomorphism but only countably many f.p. groups.

Example 4.6

1) Finite groups are f.p.

2) \( C_n = \langle x \mid x^n \rangle \) (but \( \langle x, y \mid x^{14}, y^{21}, xy^3 = y^4 \rangle = C_7 \))

Proposition 4.7

If \( N = \langle n_i \mid r_i \rangle, i \in I, i \in J, H = \langle h \mid l \rangle \), then \( G = N \times H \) has presentation \( P = \langle n_i, h, r_i, s_i, h_n : h_r^{-1} = \varphi(h_r)(\lambda_i) \rangle \)

Proof

As \( h_n : = (\text{element of } N) h_r \), any \( p \in P \) can be written as \( v(h_n), w(h_r) \) (words in various \( n_i, h_r \)). Define \( \Theta: P \rightarrow G \) via \( F : \{ n_i \mid r_i \} \rightarrow G \) for \( p \in P \). If \( \Theta(p) = e \) then \( \Theta(v(h_n)) \Theta(w(h_r)) = e \) \( \forall n \in N, h \in H \). But \( N \cap H = I \) so \( v(h_n), w(h_r) = e \in P \). \( e \) injective on \( N, H \) separately.

\( \therefore \) \( \Theta \) is an isomorphism.

If \( G = \langle x_1, \ldots, x_n, r_1, r_2, \ldots \rangle \) and \( p \) prime.

Let \( F_p \in \mathbb{F}_p \) be the exponential sum vector mod \( p \) of \( \lambda_i \). \( F_p = \begin{pmatrix} x_1 s^p & x_2 s^p & \cdots & x_n s^p \\ x_1 s^p & x_2 s^p & \cdots & x_n s^p \\ \vdots & \vdots & \ddots & \vdots \\ x_1 s^p & x_2 s^p & \cdots & x_n s^p \end{pmatrix} \mod p. \ F : \langle p \rangle \rightarrow \mathbb{F}_p \) is a homomorphism.
Proposition 4.8

If \( S = \text{span} \{ \tilde{x}_1 \} \neq (F_p)^n \) then \( G \neq 1 \).

Proof

Consider \( S \neq (F_p)^n \), could take \( a \in S \). Let \( f : x \mapsto a \cdot x \).

If \( f \) is linear \( f : (F_p)^n \rightarrow F_p \) with \( S \subseteq \ker(f) \).

So \( f_n : (F_p)^n \rightarrow F_p \) factors through \( G \).

So \( G \rightarrow F_p \).

Example 4.9

If \( \# \text{relators} < \# \text{generators} \) in a finite presentation, then the group \( G \rightarrow F_p \), so it is infinite.

Proposition 4.10

If \( G = \langle x_i \mid r_c \rangle \), \( H = \langle y_i \mid s_c \rangle \), then \( G * H = \langle x_i, y_i \mid r_c, s_c \rangle = \text{RHS} \).

Proof

Let \( \text{RHS} = G / H \rightarrow G * H \) \( (\Theta \text{ fixes } x_i, y_i) \).

Also, \( \exists \) a homomorphism from \( G \) into \( G / H \) fixing each \( x_i \) and from \( H \). Extend to \( \Phi : G * H \rightarrow G / H \) by 3-20. Now compose with \( \Theta \), and extend.

Uniqueness says \( \Theta \Phi = \text{id} \) but \( \Phi \) is surjective.

Nielsen-Schreier

\[ \text{Free} \rightarrow F_p \]

Subgroups \( \times \rightarrow (F_2) \)

Quotients \( \times \rightarrow (4.5) \)

Extensions \( \times \rightarrow \)
Theorem 4.11 (P. Hall)

If \( N \trianglelefteq G \), then \( N, \frac{G}{N} \text{ f.p.} \Rightarrow G \text{ is also f.p.} \)

Proof

Let \( N = \langle x_1, \ldots, x_n, r_1, \ldots, r_m \rangle \)
\( \frac{G}{N} = \langle y_1, \ldots, y_k | s_1, \ldots, s_k \rangle \)

Take \( g_1, \ldots, g_k \) in \( G \) with \( g_i N = y_i \). So we have a generating set \( g_1, \ldots, g_k, x_1, \ldots, x_n \)

Now take relations \( r_i = e, S_i (g_1, \ldots, g_k) = s_i (x_1, \ldots, x_n) \)
\( g_i x_i g_i^{-1} = u_i (x_1, \ldots, x_n) \in N \).
\( g_i^{-1} x_i g_i = v_i (x_1, \ldots, x_n) \in N \).

\( \square \)

continues later on
\( G \) a group.
\( A, B \leq G \)
\( \varphi : A \to B \) an isomorphism

HNN extension:
\[
G \ast \langle t \rangle = \langle \langle t \rangle \rangle
\]

i) Free Product:
\[
\{ G_{\lambda} : \lambda \in \Lambda \} \text{ an indexed family of groups}
\]

ii) Reduced Sequence in \( \{ G_{\lambda} \} \):

A finite sequence \( g_1, \ldots, g_n \) where \( g_i \in \prod_{\lambda \in \Lambda} G_{\lambda} \),
\( g_i \neq e \), no successive \( g_i, g_{i+1} \) are in the same \( G_{\lambda} \).

\( R = \{ \text{reduced sequences} \} \)

\( \ast \) \( G_{\lambda} \leq S(R) \)

generated by elements \( \tau(g, \lambda) \)

Here, \( \lambda \in \Lambda \), \( g \in G_{\lambda} \setminus I \)

\[
\tau(g, \lambda)(g_1 \ldots g_n) = \begin{cases} 
(g, \lambda)g_1 \ldots g_n & \text{if } g_i \not\in G_{\lambda} \\
((g, \lambda)g_1)g_2 \ldots g_n & \text{if } g \in G_{\lambda}
\end{cases}
\]
Theorem 4.11 (continued)

Take \( G = \langle x_1, \ldots, x_n, \bar{x}_1, \ldots, \bar{g}_1 \rangle \) with the above relations. Then \( G \cong G \), so let \( K = \ker \Theta \), and \( N = \langle \bar{x}_1, \ldots, \bar{x}_n \rangle \triangleleft G \) by \( \psi _1, \psi _2 \).

Now restrict \( \Theta \) to \( N \) where it is an isomorphism. So \( K \cap \bar{N} = 1 \).

Now, as \( \Theta (\bar{N}) = N \), we obtain \( \Theta _0 : \bar{N} \cong \bar{N} \) which is also 1-1 as above, but \( \ker \Theta _0 = \bar{K} \bar{N} / \bar{N} \cong \bar{K} \bar{N} / \bar{N} \), so \( K = 1 \).

Free Products with Amalgamation and HNN-extension

Definition 4.12

Let \( G \) and \( H \) be groups with \( A \leq G \), \( B \leq H \), such that \( \Phi : A \to B \) is an isomorphism. The free product with amalgamation is the group \( G * _{A=\Phi (A)} H \) or a generating set for \( A \).

Definition 4.13

If \( G \) is a group with \( A, B \leq G \) and \( \Psi : A \to B \) is an isomorphism then the HNN extension \( G * \langle t \rangle \) is the group \( G * \langle t \rangle \langle t a t^{-1} = \Phi (a) \rangle \).

Note that \( t a = \Phi (a) t \), \( t^{-1} b = t^{-1} \Phi (b) t^{-1} \), so we can move \( a \) as \( (a b) \) to the left of \( t \)'s \((t^{-1})\) in \( G * \).

Choose right transversals \( T_A \) and \( T_B \) for \( A \) and \( B \) in \( G \), both including \( e \). A normal form is a sequence \( g_1 t^e_1 g_1 t^e_2 \ldots t^e_n g_1 \) (for \( n \geq 0 \), \( E_i = \pm 1 \)) such that each \( g_i \in G \) with \( E_i = \pm 1 \) \( \Rightarrow g_i \in T_A \) and there is no subsequence \( t^e \).
Every element of $\mathcal{G} \times \mathcal{Q}$ can be put into normal form (work from right to left).

Theorem 4.14 (Normal form for HNNs)

Every element in $\mathcal{G} \times \mathcal{Q}$ has a unique normal form.

Define $\rho : \mathcal{G} \times \mathcal{Q} \rightarrow S(\mathcal{N})$ by:

i) $\rho((g, t) \cdot t_0^e \cdots t_n^e) = g \cdot t_0^e \cdots t_n^e$ for $g \in \mathcal{G}$.

Note that for $g, h \in \mathcal{G}$, $\rho(g \cdot h) = \rho(gh)$ so $\rho(g) \in S(\mathcal{N})$.

ii) If $e_i = -1$ and $g_0 \in A$ then

$\rho((t) \cdot t_0^e \cdots t_n^e) = \rho((g_0 \cdot t) \cdot t_0^e \cdots t_n^e)$

Otherwise $\rho((t) \cdot t_0^e \cdots t_n^e) = \rho((a) \cdot t_0^e \cdots t_n^e)$

where $g_0 = a \cdot t_0$ for $g_0 \in T_a$.

This has inverse

$\rho(t) \cdot t_0^e \cdots t_n^e = \rho((g_0 \cdot t) \cdot t_0^e \cdots t_n^e)$

otherwise $g_0 \cdot t_0^e \cdots t_n^e 1 \rightarrow \rho((a) \cdot t_0^e \cdots t_n^e)$

Also $\rho(a) = \rho(t) \cdot \rho((a)) \cdot \rho(t)$ so $\rho$ is a well-defined homomorphism on $\mathcal{G} \times \mathcal{Q}$ and $\rho$ is injective hence an isomorphism.

We say that $g_0 \cdot t_0^e \cdots t_n^e$ is reduced if there is no subsequence $t g \cdot t_i^e \cdots t_n^e$ for $g \in A$ or $t \cdot g \cdot t_i^e \cdots t_n^e$ for $g \in B$.

Reduced sequence need not have $e_i = e_i$. 

\[ \frac{1}{2} \]
Corollary 4.15 (Britton's Lemma)

$G$ embeds in $G \ast \Phi$ by $g \mapsto g$ and if for $n \geq 1$, $g_0 \ldots g_n$ is reduced, then it is $\neq e$ in $G \ast \Phi$.

Proof

$g(\neq e), e$ are both in normal form, so $g \neq e$ in $G \ast \Phi$ by 4.14. On changing a reduced sequence into a normal one, no $t^{\pm 1}$ cancel and

$\phi$ normals from $g_0 \in \Phi \ldots g_n$} \[ \boxed{\square} \]

Corollary 4.16 (Torsion in HNN)

If $r \in G \ast \Phi$ has finite order, then $r$ is conjugate to some $g \in G$.

Proof

If $r = g_0 t^{E_1} \ldots g_n (n \geq 1)$ is a reduced sequence and $t^{E_n} g_n g_0 t^{E_1}$ is not a pinch, then $r^k$ is reduced and so is $\neq e$ by 4.15. Otherwise, replace $r$ by $t^{E_n} g_n r (t^{E_n} g_n)^{-1} = g_0 t^{E_2} \ldots t^{E_n} g_n \ldots$, which is either reduced, or in $G$. Now repeat. \[ \boxed{\square} \]

Inve 2

If an infinite group where every element (except $e$) is conjugate.

Proof

Take $G$ to be countably infinite and torsion free (e.g. $\mathbb{Z}$).

Let $[g_0, g_1, \ldots]$ be an enumeration of non-id elements of $G$.

We form the following HNN extension:

$G_1 = \langle G, t, \mid t_i g_0 t_i^{-1} = g_i \rangle \quad (<g_i> = \mathbb{Z})$

$G_2 = \langle G_1, t_2, \mid t_2 g_0 t_2^{-1} = g_2 \rangle$
$G$ embeds in $G_i$ by 4.15, so we still have infinite order.
continue. We obtain $G \leq G_i \leq \ldots$, an ascending sequence.
Now let $\Gamma_i(G) = \cup_i G_i$. Then $\Gamma_i$ is countably infinite, and
torsion free by 4.16.
Note that any 2 (non-identity) elements of $G$ are conjugate in
$\Gamma_i(G) = \Gamma_i$.
Now set $\Gamma_1 = \Gamma$ ($\Gamma_i$), $\ldots$, $\Gamma_m = \Gamma(\Gamma_m)$ and form
$\Delta = \cup_m \Gamma_m$. Now, $\gamma, \delta \in \Delta \Rightarrow \gamma, \delta \in \Gamma_m$, so
are conjugate in $\Gamma_m$ and so in $\Delta$.

(D Ohm, 2010, E.g. examples. For example, open problem)
For free products with amalgamation, $G \ast H$, we say an element
$g_1 \ldots g_n \in G \ast H$ is a-reduced if for $n > 1$, no $C_i$ is in
$A$ or $B$. We can turn any $g \in G \ast H$ into a-a-reduced
element by "absorbing" elements of $A$ or $B$.

**Corollary 4.17**
If $g_1, \ldots, g_n$ is a-reduced, then it is non-identity in $G \ast H$
($G, H \rightarrow G \ast H$)

**Proof**
Let $F = (G \ast H) \ast \langle t \rangle$ \((\text{cat}^{-1} = \{a\})\), HNN extension of $G \ast H$.
Let $\Psi : G \ast H \rightarrow F$ be defined by: $\Psi(a) = t g t^{-1}$, $\Psi(h) = h$.
In $G \ast H$, $a = \Psi(a)$, $g(a) = (t g t^{-1})$, $\text{cat}^{-1} = \{a\}$ in $F$.
This is well defined: for a-reduced element, if $C_i \in A$
then $g_i \rightarrow \Psi(a) \neq 1$ (normal form).
Otherwise, $W_i \rightarrow$ reduced sequence in $F$ (HNN) so done
by 4.15
Definition 5.1

The group $G$ is soluble (solvable) if there exists a sequence
$I = G_n \triangleleft G_{n-1} \triangleleft \ldots \triangleleft G_1 \triangleleft G_0 = G$, with $G_i/_{G_{i+1}}$ abelian.

Theorem 5.2

This property is preserved by subgroups, quotients and extensions.

Proof

If $H \triangleleft G$, $G$ soluble, $G_i \triangleleft G$, $G_i/_{G_{i+1}}$ abelian,
and $(H \cap G_i)/_{(H \cap G_{i+1})} \triangleleft G_i/_{G_{i+1}}$ abelian,
then $G_i/_{G_{i+1}}$ with quotient $G_i/_{G_{i+1}} N \triangleleft G_i/_{G_{i+1}} N$ abelian.

For any group $G$ the derived series is the sequence
of subgroups $G = G^{(0)} \triangleright G^{(1)} \triangleright \ldots$
for $G^{(i+1)} = (G^{(i)})'$ derived or commutator subgroup.
$G^{(i+1)}$ is characteristic in $G^{(i)}$. 

Definition 5.3

Take $p'$ of sequence for $G^{(i)}$ to get $G^{(i)}_1$.
Then use sequence for $N$. 

1
Proposition 5.4

$G$ is solvable $\iff$ derived series terminates at $1$. If so then it has the same length of a smallest series with abelian quotients.

$(\Leftarrow)$ is clear.

Proofs

$G^{(i+1)} = [G^{(i)}, G^{(i)}]$, so the quotient is abelian.

Now let $G_i$ be as in (5.1) and assume that $G^{(i)} \leq G_i$. As $H \leq G \Rightarrow H^* \leq G$, we have $G^{(ii)} \leq G_i'$, but $G^{(ii)} / G_i'$ is abelian, so $(G_i')' \leq G_{i+1}$.

So in 5.1, there does exist a series where $G_i \triangleleft G$.

We say that $G$ is perfect if $G = G'$. Then $G(\neq I)$ is not solvable, and if $G \rightarrow Q \rightarrow A$ abelian, then $A = I \Rightarrow Q$ is also perfect.

e.g. $G$ simple, non-abelian is perfect, like $A_5$.

Corollary 5.5

If $G$ contains a non-abelian free subgroup, then $G$ is not solvable.

Proofs

If $G$ is solvable, $F_2 \leq G$, $F_2 / N \cong A_5$, solvable.
Topics in Infinite Groups

Polycyclic Groups

Definition 5.6

The group \( G \) is poly cyclic if there exists a chain:

\[ \{ e \} < G_1 < G_2 < \ldots < G_n = G \text{ s.t. } G_i \text{ is cyclic.} \]

Theorem 5.7

This is preserved by subgroups, quotients, extensions.

Proof

Exactly the same for as 5.2 with abelian cyclic. \( \Box \)

In fact, if a property \( P \) is preserved under subgroups and quotients, then 5.2 shows that "poly - \( P \)" has all these.

Soluble is "poly - abelian"

Corollary 5.8

F.g. abelian groups are poly cyclic.

Proof

Express \( A \) as a direct product of cyclic groups. \( \Box \)

via Structure Theorem

Theorem 5.9

\( G \) poly cyclic \( \iff \) \( G \) soluble and has max

Proof

(\( \Rightarrow \)) A cyclic group is soluble, has max, and these are preserved by extensions.
Each $G_i$ in (5.1) is f.g. so each $\frac{G_i}{G_{i+1}}$ is poly-cyclic by 5.8. This is preserved by extensions. 

Corollary

$G$ poly-cyclic, $H \leq G \Rightarrow H$ is f.p.

Proof

A cyclic group is f.p. and this is preserved by extensions \cite{4.11}.

$\Rightarrow$ $G$ is f.p. $H \leq G \Rightarrow H$ poly-cyclic.

$\Rightarrow$ $H$ is f.p. by same reasoning as $G$.

Extended Example 5.11

Let $D \leq \mathbb{Q}$ be the dyadic rationals $\left\{ \frac{n}{2^i} : n \in \mathbb{Z}, i \geq 0 \right\}$. This is not cyclic, hence infinitely generated as $\mathbb{Q}$ is locally cyclic.

Let $B = D \times \mathbb{Z}$ where $\varphi(t)$ is the automorphism $d \mapsto 2d$ of $D$. Then $B$ is abelian and generated by $1, d \in D$ and $t \in \mathbb{Z}$. \cite{1} \cite{2}.

All elements of $B$ have the form $(\frac{n}{2^i}, t \mathbb{Z})$, multiplicative with $(0, t^k)(\frac{n}{2^i}, 1)(0, e^k) = (2^{k} \frac{n}{2^i}, 1)$.

So $D \leq \langle 1, t \rangle$ by taking $k = 0$.

What about a finitely presented example?

Let $G = \langle a, b | bab^{-1} = a^2 \rangle$.

Then $a \mapsto (1, 1)$, $b \mapsto (0, t)$ extends to a homomorphism $G \rightarrow B$ which is surjective.

We show that this is injective.
Now $C' \cong \mathbb{Z}$, generated by $bc'$. As $ba = a^2 b$ (and $ba^{-1} = a^{-2} b$) and $ab^{-1} = b^{-1} a^2$ (and $a^{-1} b^{-1} = b^{-1} a^2$), then given any word in $a, b$, we can move the power of $b$ past $a^\pm 1$ to the right and $b^{-1}$ to the left.

Thus any $g \in G$ has the form $b^{-m} a^l b^n$ for $l \in \mathbb{Z}$, $m, n \geq 0$. Now $\theta(g) = (\frac{l}{2^m}, \frac{n}{2^m})$, and if $\theta(g) = e \in (0, i)$, then $l = 0$, $n = m$.

$\Rightarrow \theta$ injective, \[ \Rightarrow G \cong \mathbb{Z} \times \mathbb{Z} \]

This can also be done with matrices. Set $a = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $b = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $a, b \in \text{GL}_2(\mathbb{Z})$.

Then $b^{-m} a^l b^n = \begin{pmatrix} 2^{-m} & 2^{-m} \\ 0 & 1 \end{pmatrix}$ and $ba b^{-1} = a^2$.

**Proposition 5.12**

We have infinite f.p. groups $G$ with f.p $H \leq G$ and $g \in G \setminus H$ s.t. $g H g^{-1} < H$.

**Proof**

Take $G$ as above with $H = \langle a \rangle$. Then $6 H b^{-1} = \langle a^2 \rangle$ \[ \Rightarrow 6 H b^{-1} < H \]

**Note**

In $G$ we have $\cdots < g^2 H g^{-2} < g H g^{-1} < H < g^{-1} H g < \cdots$ Any example in 5.12 cannot have min. or max.
Nilpotent Groups

Definition
The group $G$ is nilpotent (nil) if $I$ is a chain
$I = G_n \leq G_{n-1} \leq \ldots \leq G_0 = G$ such that $G_i \leq G$
and $\frac{G_i}{G_{i+1}}$ is in the centre of $\frac{G_{i+1}}{G_i}$ (a central series).

Note
So $\frac{G_i}{G_{i+1}}$ is in $\text{centre}$ of $\frac{G_{i+1}}{G_i}$

Ab $\Rightarrow$ Nil $\Rightarrow$ Soluble but $G(\neq I)$ nilpotent $\Rightarrow Z(G) \neq I$

So $S_3$ is poly cyclic but not nilpotent. ... nil is not preserved

by extensions.

Theorem 5.14

Nil is preserved by subgroups, quotients, and direct products.

Proof
i) As in 5.2), we want $\frac{H \cap G_i}{H \cap G_{i+1}}$ is in the centre of $\frac{H}{H \cap G_i}
\frac{H \cap G_i}{H \cap G_{i+1}}$. The former becomes $\frac{H \cap G_i}{G_{i+1}}
\frac{H \cap G_i}{G_{i+1}}$, which is in $\frac{G_i}{G_{i+1}}$, so commutes with $\frac{G_i}{G_{i+1}}$, so commutes with $\frac{H \cap G_i}{G_{i+1}}$

Quotient
ii) We need $G_i \cap N$ in the centre of $\frac{G_i}{G_{i+1}} N$ but

$[x, g] = x g x^{-1} g^{-1} \in G_{i+1}$ for $x \in G_i$, $g \in G$

$[x, g] = x g (x n)^{-1} g^{-1} \in G_i \cap N$

Product
iii) Given $G_i \leq G$, $H_i \leq H$ as in 5.13, $G_i \times H \leq G \times H$

with $G_i \times H_{G_i \times H}$, so we use

$I \times I = I \times H_m \leq I \times H_{m-1} \leq \ldots \leq I \times H = G_n \times H$

$\leq G_{n-1} \times H \leq \ldots \leq G \times H$

$\square$
For $H_1, \ldots, H_n \leq G$, define $[H_1, H_2] = \langle [h_1, h_2] \rangle$ and $[H_1, \ldots, H_n] = \langle [H_1, \ldots, H_{n-1}], H_n \rangle$.

For any $G$, the lower central series of $G$ is

$$
\gamma_1 G = G > \gamma_2 G > \gamma_3 G > \cdots > \gamma_{n+1} G > \cdots
$$

Lemma 5.15

$G$ is nilpotent $\iff$ lower central series terminates at $1$.

Proofs

clear for $\gamma_2 G$, follows by induction

$(\Leftarrow)$ $\gamma_i G$ is characteristic in $G$ and $\gamma_i \frac{G}{\gamma_i G}$ is in $Z\left(\frac{G}{\gamma_i G}\right)$

$(\Rightarrow)$ Assume $\gamma_i G \leq G_{i-1}$ for $i \geq 1$. Then $[\gamma_i G, G] \leq [G_{i-1}, G]$ which (as a central series) is in $G_i$. Hence $\gamma_i G \leq G_{i-1} \forall i$

Hence the result.

Theorem 5.16 (Baer)

If $G$ is nil and $G_1 G$, then $\gamma_i G$ is finitely generated.

Proof

If $\gamma_3 G = 1$, and $G = \langle x_1, \ldots, x_n \rangle$ is symmetric (closed under inverses) then, as we have

$$
[y, x] x_1 [y, z] x_1^{-1} = [y, x z] \quad \text{and} \quad z [x, y] z^{-1} [z, y] = [2x, y]
$$

in any group, then in $G$ we get

$$
[g, xo] = [g, x] [g, o] \quad \text{and} \quad [gx, y], \quad \text{so any element of $G$ is a finite product}
$$

of the generators $\langle x_1, \ldots, x_n \rangle$.\[2\]
This also shows that for any $G$, $\frac{r_2^G}{r_1^G}$ is given by
$$\left\{ \left[ x_i, x_r^G \right] \right\}.$$ Now assume that anything in $\frac{r_2^G}{r_1^G}$ is a product of $\left\{ \left[ x_i, \ldots, x_j \right] \right\} \equiv [E]$.

In $\frac{r_1^G}{r_2^G}$, suppose that $g = x_i h$. Then
$$\left[ \left[ x_i, g \right] \right] = \left[ \left[ x_i, x_i h \right] \right] = \left[ \left[ x_i, x_r^G \right] \left[ x_i, h \right] \right],$$ as $\left[ \ldots \right]$ is in the centre.

So for $\left[ g_i, \ldots, g_i + R \right] \in r_1^G$, we can write it as
$$\prod \left[ \left[ g_i, \ldots, g_i + R \right], x_r \right] \prod \left[ \left[ x_i, \beta, x_r \right] \right] \prod \left[ \left[ x_i, x_r \right] \right] \prod \left[ \left[ x_r, R \right] \right] \prod \left[ \left[ x_i, x_r \right] \right] \prod \left[ \left[ x_r, R \right] \right] \prod \left[ \left[ x_i, x_r \right] \right] \equiv \left[ E \right].$$

Because $\left[ \beta, x_r \right] \left[ \left[ x_i, x_r \right] \right] \equiv \left[ \beta, x_r \right] \left[ \left[ x_i, x_r \right] \right]$ modulo $r_1^G$.

So if $R_{\text{max}} G = I$, then $(\exists n \geq 1)$ and $\frac{r_n^G}{r_{n-1}^G}$ f.g.

Corollary 5.17

All f.g. nilpotent groups are polycyclic (and have max)

Proof:

$\frac{r_1^G}{r_1^G}$ is abelian and f.g., so extensions have max and are polycyclic.

6 Finite Index Subgroups and Virtual Properties

If $H \leq G$ has finite index (i.e. a finite number of cosets), we write $H \leq \pi G$ with index $[G : H]$. 

\[ \]
Lemma 6.1

i) If $H \leq G$, $H \leq \mathfrak{S} \leq G$, then $H \leq \mathfrak{S} \leq G$

ii) If $\mathfrak{S} \leq H \leq G$, then $\mathfrak{S} \leq G$ with $[G: \mathfrak{S}] = [G:H][\mathfrak{S}:H]$.

iii) If $H \leq G$, $\mathfrak{S} \leq G$, then $H \cap \mathfrak{S} \leq \mathfrak{S}$ with index $\mathfrak{S} \leq [G:H]$, with equality if $SH = G$ and it divides $[G:H]$ if $SH \leq G$.

iv) If $H \leq G$, $\mathfrak{S} \leq G$, then $H \cap \mathfrak{S} \leq G$ with $[G:H\mathfrak{S}] \leq [G:H][G:J]$

Proof:

i) Suppose $g_i, i \in I$, is a (left) transversal for $H$ in $G$ and $H_i, i \in I$, for $\mathfrak{S}$ in $H$. Then $G = \bigcup_i g_iH_i$.

ii) Check $g_i H_i \cap \mathfrak{S} = g_i H_i \cap \mathfrak{S}$, then they are the same coset.

iii) Take $G = g_i H \ldots u g_k H$, and throw away any coset $g_i H$ with $(g_i H) \cap \mathfrak{S} = \emptyset$ (this happens if $SH \neq G$).

Then $g_i (H \cap \mathfrak{S}), \ldots, g_k (H \cap \mathfrak{S})$ are disjoint. Since $g_i H$ are disjoint, check that their union $U$ is also required. Replace $g_i$ with $S_iS$ as above to form $S_i (H \cap \mathfrak{S})$.

Then $U = S$ and if $s \in g_i H$ then $s = s_i h$ and $S_i s \subseteq S_i (H \cap \mathfrak{S})$.

If $SH \leq G$ (subgroup) then $H \leq SH \leq G$ (use i), ii), and we have shown that $[SH:H] = [S:G] [H:S]$

$[g:H\mathfrak{S}] = [G:J][J:H\mathfrak{S}] \leq [G:J][G:H]$
Theorem 6.2 (Principle)
A finite intersection of f.i. subgroups has f.i.

Lemma 6.3
If $[G:H] = k$ then for any $g \in G$, $F_i$ with $1 \leq i \leq k$ such that $g^i \in H$. If $H \leq G$ then we can take $i|k$ (or even $i = k$).

Proof
$H, gH, \ldots, g^kH$ cannot be distinct cosets so $g^iH = g^jH$ for $0 \leq i < j \leq k$. Then $g^{j-i} \in H$ for $1 \leq j-i \leq k$.
If $H \leq G$ then $gH$ has order dividing $k = [G:H]$.

Proposition 6.4
Let $\Theta : G \rightarrow H$ be a surjective homomorphism.

i) If $B \leq H$ then $\Theta^{-1}(B) \leq G$; in fact $[G : \Theta^{-1}(B)] = [H : B]$.

ii) If $A \leq G$ then $\Theta(A) \leq H$ with $[H : \Theta(A)] = [G : A]$.

Proof
i) We have $H = h_1B \cup \ldots \cup h_kB$, and we take $\Theta(g_i) = h_i$ to get a transversal for $C = \Theta^{-1}(B)$ in $G$. For $g \in G$, say $\Theta(g) = h_i b$. Then $g^{-1} g \in C$ and pullback sends disjoint sets to disjoint sets with $\Theta^{-1}(h_i B) = g_i C$.

ii) $\Theta^{-1}(\Theta(A)) = KA \leq G$ for $K = \ker \Theta$ with $[G : KA] = [H : \Theta(A)]$ by i), and $[G : KA] = [G : A]$.

Regular Representation
Any group acts on itself by (left) multiplication.
Now let $H$ be any subgroup and $L$ the set of left cosets.
of $H$ in $G$. The (left) regular representation $\rho$ of $G$ on $H$ is the action of $G$ given by $\rho(g)(xH) = gxH$. Note that $\text{Orb}(H) = 2$ and the stabilizer of $H \in \mathcal{H}$ is $H \leq G$.

**Lemma 6.5**

$p : G \to \mathcal{H}$

$\ker \rho = \bigcap_{x \in G} xHx^{-1}$.

**Proof**

$xH = gxH \forall x \in G \iff xgx^{-1} \in H \forall x \in G$. 

**Definition 6.6**

For $H \leq G$, the core of $H$ in $G$ is $\ker \rho$.

**Proposition 6.7**

Core $H \triangleleft G$ is the largest normal subgroup of $G$ that is contained in $H$.

**Proof**

If $N \triangleleft G$ and $N \leq H$, then $x^{-1}Nx \leq H \forall x \in G$.

**Theorem 6.8 (Useful!)**

If $H \leq G$ with $[G : H] = n$, then $\exists N \triangleleft G$ with $N \leq H$, and $[G : N] \mid n!$.

**Proof**

$p : G \to \text{coact of } H$

$H$ has $n$ elements, so $p : G \to S(n)$.

$|\ker \rho| = |\text{Im}(\rho)|$ which divides $|S(n)| = n!$. 

$N = \ker \rho$
Theorem 6.9

If \( G \) is finitely generated then for any \( n \in \mathbb{N} \), \( G \) only finitely many subgroups of index \( n \) in \( G \).

Proof

If \( G = \langle g_1, \ldots, g_k \rangle \) then \( G \) only finitely many homomorphisms \( \Theta: G \to S(n) \) as \( \Theta \) would be determined by \( \Theta(g_i) \), so only finitely many which are transitive on \( \{1, \ldots, n\} \), where \( \text{Stab}(1) \) has index \( n \) (Orbit-Stabilizer Theorem). Now, say \( H \leq G \) with \( |G:H| = n \). Then, on ordering \( H \) as \( \{H, \ldots, 3\} \), we have the regular representation of \( G \) is a transitive homomorphism from \( G \) to \( S(n) \) with \( \text{Stab}(1) = H \). \( \square \)

Corollary 6.10

If \( G \) is f.g and \( H \leq G \), then \( \exists \) \( C \) characteristic in \( G \) with \( C \leq f H \leq G \).

Proof

Let \( C = \bigcap_{\alpha \in \text{Aut}(G)} \alpha(H) \), then \( H \) and \( \alpha(H) \) have the same index in \( G \) by 6.4, with \( C \leq f C \) by 6.9 and 6.2.

If \( \beta \in \text{Aut}(G) \), then \( \beta(C) = \bigcap_{\alpha \in \text{Aut}(G)} \beta \alpha(H) \).

\[ \beta(C) = \bigcap_{\alpha \in \text{Aut}(G)} \beta \alpha(H) \]

Théorem 6.11

If \( H \leq f G \), then \( G \) f.g. f.p \( \iff \) \( H \) f.g. f.p.

Proof

\( G \) f.g means that \( \exists \Theta: F_k \to G \). Now \( H \leq f G \).
\( \Rightarrow \theta^{-1}(H) \leq F_K \) by \( 6.4 \), so it is \( F_K \) by the Nielsen-Schreier Index Theorem. so \( H \) is f.g.

Now restrict \( \theta \) to \( F_K \to H \).

Now, suppose \( G = F_K/N \) is f.p., so \( N = \langle \langle r, \ldots, r_m \rangle \rangle_{F_K} \)

Then \( H = F_K/N \) for \( N \leq F_K \leq F_K \) by the above.

Take a right transversal \( t_1, \ldots, t_n \) for \( F_K \) in \( F_K \) where \( n = [G : H] \)

By 1.18, \( N \) consists of all elements of the form

\[ (g, r_{i_1}^{-1}s_{i_1}g_1r_{i_1}^s \cdots r_{i_n}^{-1}s_{i_n}g_n) \]

for \( g_1, \ldots, g_n \in F_K \). These need not be in \( F_K \). But as any \( g \in F_K \) is \( h'w' \) for \( h, w \in F_K \), we do have the normal closure of \( \{ t_1 r_1 t_1^{-1} s_1, \ldots, t_n r_n t_n^{-1} s_n \mid 1 \leq i \leq m, 1 \leq j \leq n \} \)

in \( F_K \) is \( N \). \( \Rightarrow \) Hence \( H \) is f.p. with these relations

Finally, f.g. (resp f.p.) groups are preserved by extensions

(1.29, 4.11) so if \( H \) is f.g. (resp f.p.) then take \( N \triangleleft G \) with \( N \leq H \) by 6.8. \( N \) is f.g. (resp f.p.) \( \Rightarrow \)

\[ G/N \text{ f.g. (resp f.p.)} \]

so certainly f.g. f.p.

Note

If \( G = \langle k \text{ generators} | m \text{ relations} \rangle \)

then we say that the deficiency \( \text{def} (\text{presentation for } G) = k - m \)

\( (\geq 0 \Rightarrow G \text{ infinite}) \)

6.11 has shown that if \( \exists H \) with \( [G : H] = n \), then \( H \) has a presentation with \( \text{def} = n (k - 1 - m) + 1 \).

So \( \text{def} (\text{presentation for } H) - 1 = [G : H] (\text{def} (\text{presentation for } G) - 1) \)
How do we know a group has proper finite index subgroups? e.g. If $H < G$ with index $n$, then $\forall g \in G$, 
$q = n \left( \frac{g}{n} \right) \in H$ (6.3).

**Theorem 6.12 (Higman, 1951)**

The group $G = \langle a, \ldots, a_n | a_{n+1} a_{n+2} \cdots a_{2n-2} a_{2n-1} a_{2n} = a_2, \ldots, a_{2n} a_{2n+1} a_{2n+2} \cdots a_{3n-2} a_{3n-1} a_{3n} = a_1 \rangle$ has no finite index subgroups.

**Proof**

If $H < G$, then 6.8 gives a non-trivial finite quotient $G/H$.

Note that for $n > 1$, and a prime $p | 2^n - 1$, the least prime factor of $n$ is $< p$.

Take $r$ the order of 2 mod $p$, then $r | n, p - 1$ ($r \neq 1$) by Fermat's Little Theorem. Now say $r_1$ is the order of $a_i$ in $G/H$.

Then $a_1^{r_1} a_2 a_3 a_4 a_5 a_6 a_7 a_8 \cdots a_{2n-2} a_{2n-1} a_{2n} = a_2, \ldots, a_{2n} a_{2n+1} a_{2n+2} \cdots a_{3n-2} a_{3n-1} a_{3n} = a_1$.

Let $p$ be the smallest prime that divides $n_1 n_2 n_3 n_4$.

WLOG, $p | n_2$. Then $n_2$ has a smaller prime factor unless $n_1 n_2 n_3 n_4 = 1$.

So $p | n_2 | 2^n - 1$

Then can show $n_2$ has a smaller prime factor from first part.
Proposition 6.13

\( G \text{ in 6.12 is infinite.} \)

**Proof**
\[
G = \langle a_1, a_2, a_3, a_4 \mid a_1 a_2 a_1^{-1} = a_2, \ldots, a_4 a_1 a_4^{-1} = a_1 \rangle
\]

If \( H = \langle x, y \mid yxy^{-1} = x^2 \rangle \), \( H \) the same but with dots everywhere \((x', y', etc.)\) then \( x, y \) have infinite order by 5.11, so \( H \cong H' \), where \( y(x) = y' \), is infinite, and is

\[
\langle x, y, z \mid yxy^{-1} = x^2, xzx^{-1} = z^2 \rangle
\]

Note that \( y, z \) freely generate \( F_2 \) by (4.17) as a word \( w(y, x) \) with powers gathered: a reduced, and therefore \( \neq e \) in \( H \). \( \exists H' \)

Now take 4 copies of \( H \). \( H_i = \langle a_i, b_i \mid b_i a_i b_i^{-1} = a_i^2 \rangle \).

Form \( H_1 \odot H_2 = \langle a_1, b_1, a_2 b_1 a_2^{-1} = a_2^2, a_2 a_1 a_2^{-1} = a_1^2 \rangle = K \), infinite with \( \langle b_1, a_2 \rangle \) free.

Similarly, \( L = H_3 \odot H_4 \) \((\text{and } 1 \leftrightarrow 3, 2 \leftrightarrow 4)\). \( \langle b_3, a_4 \rangle \cong F_2 \)

Finally, make \( K \odot L \) for \( \theta(b_1) = a_4, \theta(a_2) = b_3 \).

This is \( G \). \( \langle b_3, a_4 \rangle \cong F_2 \), no infinite

---

Virtual Properties

Here, we regard groups \( H \) and \( G \) as "basically the same" if \( H \cong G \). We say that a group property \( P \) is "ok" if when \( G \) has \( P \) and \( H \cong G \) then \( H \) has \( P \). But what about finite index supergroups: if \( H \) has \( P \) and \( H \cong G \) then \( G \) might not have \( P \).

**Definition 6.14**

The point of an "ok" property is that if \( P \) is not "ok" then virtually \( P \) is not so useful to define.

If a property \( P \) is "ok", we say that a group \( G \) is
virtually P if \( F \leq H \), \( H \leq G \), where \( H \) has \( P \).

By (6.3) this is the same as \( G \) is \( P \) by finite.

For \( P \), groups cyclic \( \Rightarrow \) abelian \( \Rightarrow \) nil \( \Rightarrow \) polycyclic \( \Rightarrow \) soluble.

Example 6.15

i) \( \mathbb{Z} \times \mathbb{Z} \) is abelian but not virtually cyclic. (Sheet 1)

every finite index subgroup in \( \mathbb{Z} \times \mathbb{Z} \)

ii) The following is a nilpotent group that is not virtually abelian.

Let \( G = \langle a, b, t \mid ab = ba, t a t^{-1} = a b, t b t^{-1} = b \rangle \) be the semidirect product \( \mathbb{Z}^2 \rtimes \mathbb{Z} \). Then \( b \in \mathbb{Z}(a) \) and \( (ab)^k = b \).

iii) The following is a polycyclic group which is not virtually nilpotent.

\[ G = \langle a, b, t \mid ab = ba, t a t^{-1} = a^2 b, t b t^{-1} = ab \rangle \]

If \( H \leq G \) with \( H \) nilpotent then take \( h \neq e \) in \( \mathbb{Z}(H) \)

and set \( h = a^k b^t \).

Word Growth * Non Examinable *

If \( G = \langle x \rangle \) for \( x \) finite, then the growth function \( r_x^{(n)}: N \rightarrow N \) is \( r_x^{(n)} = \# \{ g \in G \mid g = w(x) \} \) for \( w \), word length \( \leq n \).

If \( S = x, x^{-1} \) then \( r_x^{(n)} = 15^n \).

We say that finitely generated \( G \) has polynomial word growth if either \( x \) finite

or \( \forall \) generating set, \( \exists c, d > 0 \) s.t. \( \forall n, r_x^{(n)} \leq cn^d \).
Gromov 1981: \( G \) has polynomial word growth

\[ \Leftrightarrow \text{\( G \) is virtually nilpotent} \]

\* \* \* \n
**Virtually Polycyclic Groups**

**Theorem 6.16**

The following are equivalent:

i) \( G \) is virtually polycyclic

ii) polycyclic by finite

iii) \( \text{poly} (\mathbb{Z}) \) by finite

iv) \( \text{poly} (\mathbb{Z} \text{ or finite}) \)

**Proof**

\[ \text{fairly easy to see} \]

\[ 1 = 2 \text{ and } 3 \Rightarrow 2 \Rightarrow 4. \text{ So we prove } 4 \Rightarrow 3: \]

Suppose that \( N \) is \( \text{ poly}(\mathbb{Z} \text{ by } \mathbb{Z}) \) by finite and \( G/N \cong \mathbb{Z} \). Now \( P \trianglelefteq N \) is finitely generated so by 6.10 we have \( P_0 \trianglelefteq P \) and \( P_0 \) is characteristic in \( N \) with \( P_0 \) also \( \text{poly}(\mathbb{Z}) \) (preserved by subgroups). Now \( P_0 \trianglelefteq G \) with \( G/P_0 \cong (G/P_0)/P_0 \) so \( G \) is \( \text{poly}(\mathbb{Z}) \) by finite.

But the latter is \( \mathbb{Z} \) by finite (using 3.8).

So by 1.28, \( G \cong \text{poly}(\mathbb{Z} \text{ by } \mathbb{Z}) \) by finite.

So in general, we can push all finite factors "to the right", and we can gather, as \( (P \text{ by finite}) \text{ by finite} = P \text{ by finite}. \]

**Corollary 6.17**

Virtually polycyclic groups are preserved by subgroups, quotients, and extensions. This is the smallest class containing all
finite groups and \( \mathbb{Z} \), and all have max. and are f.p.

**Proof**

The property \( P = (\mathbb{Z} \text{ or finite}) \) is preserved by subgroups and quotients, so poly \( P \) is preserved by subgroups, quotients and extensions by 5.7. Any poly \( P \) group is contained in any class with the above properties, and max, f.p., are preserved by extensions. \( \square \)

**Virtually Soluble Groups**

**Corollary 6.18**

The virtually soluble groups with max. are exactly the virtually polycyclic groups.

**Proof**

\( G \text{ polycyclic} \iff G \text{ soluble and has max.} \)

(5.1) and (6.17) \( \iff G \text{ virtually polycyclic} \iff G \text{ polycyclic by finite} \)

Are these all of the groups with max? See last lecture. **

**Theorem 6.19**

Virtually soluble groups are preserved by subgroups, quotients and extensions.

**Proof**

i) If \( H \leq G \) with \( H \) soluble and \( S \leq G \), then \( H \cap S \leq H \cap S \).

by 6.1(iii) but \( H \cap S \leq H \) no soluble by 5.2(i), \( \implies S \) is virtually soluble.

(ii) and 5.2(ii) \( \text{ solubility preserved by quotient} \)

Let \( N, N \) be virtually soluble, and take

\( M \triangleleft N \) soluble and normal of minimal index.

Then if we have soluble \( S \triangleleft N \), we have \( S \leq M \).
as \( SM \) is soluble, normal, and has finite index. Thus \( M \) is the unique normal soluble subgroup of that index, and so is characteristic in \( N \), normal in \( G \).

because automorphisms preserve index, use 6.4
Topics in Infinite Groups (2)

Proof (continued)

iii) $Q = \frac{G}{M}$, $R = \frac{N}{M}$, $\frac{G}{N} \cong \frac{Q}{R}$.

$R$ finite and has no non-trivial soluble normal subgroups (pullback)\[\Rightarrow \text{because } M \text{ has minimal index in } N\]

Now the centraliser $C = C_{Q}(R) = \bigcap_{r \in R} C_{Q}(r)$ of $Q$

by (6.2) and Orb-Stab.

$\Rightarrow$ finite intersection of fi. groups is fi.

So the abelian group $C_{Q}R \triangleleft R$, so $C_{Q}R = 1$. Thus in $Q$,$\frac{C}{C_{Q}R} \leq \frac{Q}{R}$ virtually soluble.

$\Leftrightarrow$ because it is a normal soluble subgroup

It is not soluble because it is finite abelian.

So $\frac{G}{M} = Q$ is too. Now take $H \leq G$ with $\frac{H}{M}$ soluble

so $H$ is by 5.2 iii) $M, \frac{H}{M}$ both soluble

Corollary 6.20

Virtually soluble groups are the smallest class preserved by subgroups, quotients and extensions that contains all abelian and finite groups.

$P = (\text{Abelian or finite})$?

Proof

Like (6.17) but using (6.19)

As in (5.5), if $F_{2} \leq G$, then $G$ is not virtually soluble.

If $S \leq G$ and $F_{2} \leq G$ then $F_{2} \cap S \leq F_{2}$ with $F_{2} \cap S \leq S$ non-abelian, free, so $\ast$ by 5.5 otherwise we could show that $S$ is soluble

In Sheet 2, we have an example not containing $F_{2}$, preserved by subgroups, quotients and extensions.

Finitely generated example $\mathbb{Z} \ast \mathbb{Z}$ is not virtually soluble, no $\mathbb{F}_{2}$ subgroup.

Finitely presented examples? There are examples, but only
5 constructions.
Tits Alternating (**) Non Examinable

A finitely generated linear group (of non matrices over a field 
F) or even any linear group with char = 0 is either virtually
soluble or contains F_2.

7 Maximal (Normal) Subgroups

Definition 7.1
A proper subgroup H is maximal if \( H \leq J \leq G \)
\[ \Rightarrow H = J \text{ or } J = G. \]

Example 7.2
Q has no maximal subgroups: if \( M < Q \) for M
maximal then \( M < Q \) and \( Q/M \) has no proper non-trivial
subgroups, so is \( C_p \). But Q has no proper finite index
subgroups.

Zorn's Lemma

Poset \( X \), relation \( \leq \) (reflexive, transitive, antisymmetric)
Total order: always have \( x \leq y \) or \( y \leq x \). A subset
\( S \) of \( X \) is a chain if \( S \) is totally ordered.
Assume that if every chain \( S \) of \( X \) has an upper bound
in \( X \) (\( b \in X \) such that \( s \leq b \) \( \forall s \in S \)), then \( X \) has
maximal element \( m \) (if \( m \leq x \), then \( m = x \)).
Proposition 7.3 (Neumann, 3rd)
If $H \leq G$ and $g \in G \setminus H$ then there is a maximal subgroup $M$ containing $H$ relative to $g$, i.e. $H \leq M \leq G$ and $g \notin M$ such that if $M \leq L$, $g \notin L$, then $L = M$.

Proof

Poset = $\{ J \leq G : H \leq J \text{ and } g \notin J \}$ ordered by $\leq$.
Then for a chain $S = \{ J_i \}$ we get $U_i J_i$ a subgroup which contains $H$ but not $g$. So $\exists$ maximal $M$. \( \square \)

Corollary 7.4

Important!

If $G$ is f.g. and $H \leq G$, then $H \leq M$ for $M$ maximal.

Proof

$G = \langle g_1, \ldots, g_k, h_1, \ldots, h_r \rangle$ for $g_i \notin H$, $h_j \in H$.
Take $M_1$ maximal with $H \leq M_1$ relative to $g_1$.
If $M_1 \leq L \leq G$ then $g_1 \in L$ but not all $g_i$ are, so
WLOG, $g_2 \notin L$ and if there is no such $L$, $M_1$ is maximal so
we are already done. Take $M_2$ maximal with $L \leq M_2$ relative
to $g_2$. This must stop at or before $k$.
If $g_1, \ldots, g_{k-1} \in M_k$ maximal relative to $g_k$ then $g_k \notin L$, so $L = G$, so $M_k$ is truly maximal. \( \square \)

Note

All of this discussion works exactly the same if we replace
"subgroup" by "normal subgroup".
Infinite Simple Groups

\( G \neq \{e\} \) simple : if \( N \triangleleft G \) then \( N = I \) or \( G \).

\( G \) simple, solvable \( \implies G \cong C_p \).

Other simple groups are \( A_n \) (\( n \geq 5 \)) and \( \text{PSL}(n, F) \) for \( n > 2 \) or \( |F| > 3 \) (proof does work for infinite fields).

Example 7.5 (Irre 3)

\( H = \bigcup_{i=5}^\infty A_n \leq S(N) \) as in (i.5ii) is simple, because

if \( N \not\triangleleft H \), then \( NnA_n \ntriangleleft A_n \). If \( N \neq I \), take

\( k \) with \( NnA_k \neq I \), so \( A_n \triangleleft N \), \( \forall n \geq k \).

\( H \) is not f.g.

If \( G \) is an infinite simple group then \( G \) is not virtually soluble

as \( S \triangleleft G \implies \text{NS}\triangleleft S \) with \( N \triangleleft S \) by (6.8) so

\( N = S = G \), so \( G \) has no finite index subgroups.

Theorem 7.6 (Higman, 56)

Infinite f.g. simple groups exist.

Proof

See 6.12. \( G = \langle a_1, \ldots, a_4 \mid a_1a_2a_1^{-1} = a_2^2, \ldots, a_3a_4a_3^{-1} = a_4^2 \rangle \)

Take Higman's \( G \), and \( N \not\triangleleft G \) a maximal normal subgroup,

(containing \( I \)), \( \implies \frac{G}{N} \) simple. But \( N \neq G \), \( N \triangleleft G \)

\( \frac{G}{N} \) is infinite and f.g.

What about infinite finitely-presented simple groups?

The Thompson groups \( F, T, V \) finitely presented, infinite.

\( F \) is not simple, not virtually soluble, no \( F_2 \) subgroup.

\( T, V \) are simple.
Open Question

Does there exist an infinite f.p. simple group with no $F_2$-subgroup (Burger and Moser, 1998):

There exists $k, l > 2$ such that $G = F_k \ast_{F_l} F_k$ for $F_k \leq F_l$ is f.p., torsion-free, and simple.

8. Residual Finiteness

Definition 8.1

A group $G$ is residually finite (resp. finitely generated) if

$$\bigcap_{H \leq G} H = \{e\}.$$
Exam

Friday 30th May. Choose 3 questions from 5. Question out of 32.

Definition 8.1

$G$ is residually finite if $\bigcap_{H \leq G} H = 1$

Proposition 8.2

The following are equivalent:

i) $G$ is residually finite.

ii) $\forall \eta \exists N = \{e\}$

iii) $\forall g \in G \setminus \{e\}, \exists \text{a homomorphism } \Theta \text{ onto } F, \text{ a finite group such that } \Theta(g) \neq e.$

iv) $\forall g_1, \ldots, g_n \in G \setminus \{e\}, \exists \Theta \rightarrow F, \text{ finite, with } \Theta(g_i) \neq e.$

Proof

ii) $\Rightarrow$ i) Clear.

i) $\Rightarrow$ ii) By 6.8. "useful theorem"

ii) $\Rightarrow$ iii) as for $g \in G \setminus \{e\}, \text{ take } N = ker \Theta \text{ or } \Theta : G \rightarrow G$

where $G \neq N \rightarrow$ such an $N$ exists by (ii) with $N \leq G$

ii) $\Rightarrow$ iv) via $\Theta : G \rightarrow \bigcap_{i=1}^{n} N_i$ and Poincaré,

whereas iv) $\Rightarrow$ iii) is clear.

So finite groups and $\mathbb{Z}$ are residually finite, but $\mathbb{Q}$ and the Higman group are not.

An infinite residually finite group has infinitely many finite index subgroups (Poincaré) of arbitrarily high index by 6.11): if both $H = \bigcap_{i=1}^{n} H_n, J \leq \Theta (G)$
then \( [G : H] = [G : H \cap J] \iff H = J \) to take
\( H \cap J = J \) with \( h \in H \setminus J \), continue, and index \( \to \infty \).

*Aside*

Topological groups: Given f.g. group \( G \), we expect the discrete topology. A more interesting topology defines basic open sets to be cosets for \( N \), \( N \triangleleft G \). This is the profinite topology. Hausdorff \( \iff \) \( G \) residually finite.
Indiscrete \( \iff \) No proper finite index subgroups.
Discrete \( \iff \) \( G \) finite. **

**Lemma 8.3**

If \( R_a = \bigcap_{N \triangleleft G} N \), then \( R_a \) is residually finite.

**Proof**
Normal finite index subgroups of \( R_a \) are \( N \cap R_a \) for \( R_a \leq N \triangleleft G \), but for \( g \not\in R_a \), we have \( R_a \leq N \triangleleft G \) with \( g \not\in N \).

Note that if \( G \overset{\theta}{\to} Q \) is residually finite, then \( \theta \) factors through \( R_a \) as \( \theta(R_a) \leq R_a \).

**Proposition 8.4**

1. \( G \) residually finite, \( H \leq G \Rightarrow H \) residually finite
2. \( H \) residually finite, \( H \leq G \Rightarrow G \) residually finite
3. \( G, H \) residually finite \( \Rightarrow G \times H \) residually finite and if \( G \) is finitely generated then \( G \times H \) is residually finite.
Proof

i) \( R_s = \bigcap L \) from 8.2 and \( R_s \cap H \supseteq R_h \) by 6.(iii)

ii) For \( H \leq G \) we have \( L \leq H \supseteq L \leq G \), so \( R_a \leq R_h \).

But \( R_a \supseteq R_h \) by i).

iii) For \((g, h) \neq \text{id}\), take \( \Theta : G \to F_1 \), \( \Theta_2 : H \to F_2 \),

\( \Theta_1(g), \Theta_2(h) \) not both \( e \). Then \( \Theta \times \Theta_2 : G \times H \to F_1 \times F_2 \),

\( (g, h) \mapsto (\Theta(g), \Theta_2(h)) \), \( (g, h) \not\subseteq (e, e) \).

For \( G \times H \), we have \( \Theta : G \times H \to H \) with \( \Theta(gh) = h \), so

\( \exists \phi \) with \( \phi \Theta(gh) = \phi(h) \neq e \) in some finite \( F \), unless \( h = e \).

Now, take \( L \leq G \) with \( g \notin L \), \( g \in G \setminus \{e\} \), and \( C \leq L \), characteristic in \( G \) by 6.10. Then \( C \leq G \times H \).

Because \( H \) is residually finite.

Corollary 8.5

\( G \) virtually polycyclic \( \Rightarrow \) \( G \) residually finite.

Proof

We have \( H \leq G \) with \( H \) polycyclic by 6.16.

Now suppose that if \( M \times H = \mathbb{Z} \), \( N \) finitely generated and residually finite, then \( M \cong N \times \mathbb{Z} \) by 3.9 ii), so

\( M \) residually finite by 8.4 iii).

This \( H \) is residually finite and \( G \) is too by 8.4 iii).
Proof 1

Recall 3.23, that \( F = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \), \( G = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \) \( \in \text{SL}_2(\mathbb{Z}) \)

freely generate \( F_2 \). Given a reduced word \((\neq \emptyset)\) \( w \in F_2 \), we have \( w(F, G) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \).

Take a large prime \( p \), \( p > \max \{ |a|, |d|, |b|, |c| \} \). Then

\[ \Theta : \text{SL}_2(\mathbb{Z}) \to \text{SL}_2(\mathbb{F}_p) \] (reduction mod \( p \)).

\( \text{SL}_2(\mathbb{F}_p) \) is finite and \( \Theta(w(F, G)) \neq e \). So \( F_2 \) is

residually finite. by alternative characterisation of residual finiteness.

Now \( F_n \triangleleft F_2 \) so these are residually finite. For \( w (\neq \emptyset) \in F(x) \)

where \( x = \{ x_i : i \in I \} \), only \( x_i, \ldots, x_{i_{\text{max}}} \) appear in \( w \).

So we have \( \Theta : F(x) \to F_k = F(x_i, \ldots, x_{i_{\text{max}}}) \)

given by sending the rest of \( x \) to \( e \) and extending.

Now \( \Theta(w) \neq e \), so we now have \( \Phi : F_k \to \text{finite group} \)

with \( \Phi \Theta(w) \neq e \).

Proof 2

\( F_2 \) free on \( a, b \). We will create a reduced word \( w \) using

\( a, b, A = a^{-1}, B = b^{-1} \) (formal inverses)

\[ w = A_5 A_6 A_5 A_6 A_5 A_6 A_5 A_6 A_5 A_6 A_5 A_6 A_5 A_6 A_5 A_6 A_5 A_6 A_5 A_6 A_5 A_6 A_5 A_6 A_5 A_6 A_5 A_6 A_5 A_6 A_5 A_6 A_5 A_6 A_5 A_6 A_5 A_6 A_5 A_6 A_5 A_6 A_5 A_6 A_5 A_6 A_5 A_6 A_5 A_6 A_5 A_6 A_5 A_6 A_5 A_6 A_5 A_6 A_5 A_6 A_5 A_6 A_5 A_6 A_5 A_6 A_5 A_6 A_5 A_6 A_5 A_6 A_5 A_6 A_5 A_6 A_5 A_6 A_5 A_6 A_5 A_6 A_5 A_6 A_5 A_6 A_5 A_6 A_5 A_6 A_5 A_6 \]

Let \( \Gamma : F_2 \to S(11) \) (or \#1 if length \#1 be given by

\[
\begin{array}{cccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
\hline
a & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
6 & 7 & 8 & 9 & 10 & 11
\end{array}
\]

Partial Functions. Each \( i \mapsto i \pm 1 \).

Is this injective?
Topics in Infinite Groups

Suppose e.g. \( f(b)(a) = 10 = f(b)(11) \)?
Then "\( B^{10} b^9 \) reduced"
\[ f(a), f(b) \in S(11). \]

Now by the universal property, we can extend to a homomorphism \( f : F_2 \to S(11) \), \( f(n) (11) = 11 \), so \( f(w) \neq id \)

\[ \begin{array}{ccccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\
q & 11 & 12 & 23 & 45 & 67 & 89 & 1011 \\
6 & 11 & 31 & 24 & 15 & 67 & 10 & 29 \\
\end{array} \]
Theorem 8.6

1) \( G_1, G_2 \) residually finite \( \Rightarrow \) \( G_1 \ast G_2 \) residually finite

Proof

First, suppose \( G_1, G_2 \) are finite. Given a reduced sequence

\[ g_1 g_2 \ldots g_n \in G_1 \ast G_2 \text{ of length } n \geq 1, \]

let

\[ X_n = \{ g \in G_1 \ast G_2 \mid 0 \leq \text{length}(g) \leq n \} \]

a finite set because \( G_1, G_2 \) finite

Define an action on \( X_n \) via the following:

- If \( r \in G_2 \) then \( r(g_1, \ldots, g_k) = \begin{cases} r g_1 \ldots g_k & \text{if length (RHS) } \leq n \\ g_1 \ldots g_k & \text{otherwise (k=n, } g_i \in G_1 ) \end{cases} \]

\[ \exists \text{ a homomorphism } G_2 \rightarrow S(X_n) \]

Similarly, \( \exists \) a homomorphism \( G_1 \rightarrow S(X_n) \)

We can extend this to a homomorphism \( G_1 \ast G_2 \rightarrow S(X_n) \), a finite group. (finite because \( X_n \) is a finite set)

Now done by 8.2, definition of res. fin. with homomorphisms.

Now \( g_1 g_2 \ldots g_n \emptyset = g_1 g_2 \ldots g_n \neq \emptyset \) so that \( g_1 g_2 \ldots g_n \) is non-trivial in \( S(X_n) \) (acts non-trivially on \( \emptyset \))

For \( g_1 g_2 \ldots g_n \) in general \( G_1 \ast G_2 \), choose \( N_1, N_2 \uparrow G_1 \), \( N_2 \uparrow G_2 \) such that \( g_1, \ldots, g_n \in N_1 \cup N_2 \). By 8.2iv),

\[ G_1 \rightarrow G_1 / N_1 \leftarrow \left( G_1 / N_1 \right) \ast \left( G_2 / N_2 \right) \]

This extends to \( G_1 \ast G_2 \rightarrow \left( G_1 / N_1 \right) \ast \left( G_2 / N_2 \right) \) and the image

of \( g_1, \ldots, g_n \) is reduced so has length \( n \).

and \( g_1, \ldots, g_n \in N_1 \cup N_2 \) do not reduce to identity elements

Hopfian Groups

Definition 8.7

A group \( G \) is Hopfian if every surjective endomorphism
\[ \Theta : G \rightarrow G \text{ is injective.} \]

If not then \( \ker \Theta \neq G \), so \( G \) is isomorphic to a proper quotient of itself. Finite groups are Hopfian, \( \mathbb{Z} \) too.

The free group on \( F(\mathbb{N}) \) is not Hopfian as we send \( x_i \mapsto x_i \), \( x_i+1 \mapsto x_i \), but it is residually finite.

**Theorem 5.8 (Malcev 1940)**

A finitely generated, residually finite group \( G \) is Hopfian.

**Proof**

For \( \Theta : G \rightarrow G \) and \( H \leq F \) with index \( n \), \( \Theta^{-1}(H) \) has index \( n \) too, and if \( \Theta^{-1}(H_1) = \Theta^{-1}(H_2) \), then\[ \Theta \Theta^{-1}(H_1) = H_1 = H_2 . \]So the pullback map is injective on \( \{ \text{index } n \text{ subgroups of } G \} \). But this is a finite set by 6.9, so is a permutation. However, \( \ker \Theta \leq \Theta^{-1}(H) \)

\[ V H \leq G \text{. So } \ker \Theta \text{ is in } \bigcap_{H \leq G} \Theta^{-1}(H) = R G = 1 . \]

**Corollary 5.9**

If \( g_1, \ldots, g_n \) generate the free group \( F_n \), then they freely generate \( F_n \).

**Proof**

\[ \mathbb{Z} \text{ residually finite } \Rightarrow \mathbb{Z} \leq F_n \text{ residually finite} \]

\[ 8.8 \Rightarrow F_n \text{ Hopfian as it is also f.g.} \]

\[ F_n \text{ Hopfian, so if } w(g_1, \ldots, g_n) = e \text{ in } F(x_1, \ldots, x_n). \]

Then \( \Theta : F(g_1, \ldots, g_n) \rightarrow F(x_1, \ldots, x_n) \) by:

- given a symbol \( g_i \), \( g_i \mapsto (\text{its image on RHS} ) \), a reduced word in \( x_1, \ldots, x_n \). Extend this homomorphism.

This is injective since it hits the generating set.
Hence this is injective (8.6, 8.8), so \( n = 0 \).

An infinite simple group is Hopfian, but not residually finite.

**Braunslag–Solitar Groups**

**Definition 8.10**

The Braunslag–Solitar group \( B_{m,n} = \langle a, b \mid t a^m t^{-1} = a^n \rangle \)

for \( m, n \neq 0 \). So these include \( B_{1,1} = \mathbb{Z} \times \mathbb{Z} \),

\( B_{1,-1} = \langle \text{Fundamental Group of the Klein Bottle} \rangle \)

We can also change \( m, n \) and keep the group the same:

\( B_{m,n} \cong B_{-m,-n} \), \( B_{m,n} \cong B_{n,m} \)

They are HNN extensions \( \langle a \rangle \ast_{\phi} \) with \( \phi: \langle a^n \rangle \to \langle a^m \rangle \).

**Proposition 8.11**

\( B_{m,n} \) is soluble if \( |m| = 1 \) or \( |n| = 1 \) and contains \( F_2 \) otherwise.

**Proofs**

If \( |m| = 1 \) or \( |n| = 1 \), then WLOG we have \( B_{1,n} \), and this is soluble, just as in 5.11 \( (B_{1,2}) \). Otherwise, \( a \notin \text{domain or image of } \phi \). So for any reduced word \( w(x,y) \in F_2 \), we have \( w(t, a t a^{-1}) \) is a reduced sequence in an HNN extension. So this is \( \neq e \) by Britton's Lemma.

**Theorem 8.12**

\( B_{2,3} \) is not Hopfian.

**Proof**

Let \( \Theta(t) = t, \Theta(a) = a^2 \). This is a homomorphism as it
Consider $\mathcal{G} = \langle a \rangle$.

Proof: By Theorem 8.13, $\mathcal{G} = \langle a \rangle$ is generated by $\theta(a) = \theta$.

Now $\mathcal{G}$ is generated by $\theta(a)$, as well as $\langle e \rangle = \langle \theta(a) \rangle = \mathcal{G}$, which follows by considering $\theta(a)$ in $\mathcal{G}$, as it is in $\mathcal{G}$.

So $\mathcal{G} = \langle e \rangle = \mathcal{G}$ is not a proper subgroup of $\mathcal{G}$.

For $y \in \mathcal{G}$, $\theta(y) = e$, we have $\mathcal{G} = \langle e \rangle = \langle \theta(a) \rangle = \mathcal{G}$.

What about the kernel? $\theta([\mathcal{G}, \mathcal{G}]) = \langle a^2 \rangle = e$.

presumes the relation $\theta([a, a]) = \langle a^2 \rangle$.

and these commute because $\theta(a) = (\theta(a))^a$ for $\theta(a)$.
9. The Generalized Burnside Problem

Examples of Torsion Groups

Finite groups $F$, infinite $F \times F \times \ldots$, example 1.5.
None are f.g.

1. Generalized Burnside Problem (MOZ)

Do there exist infinite f.g. torsion groups?

Lemma 9.1

If $G$ is such a group then

i) $G \rightarrow Q \Rightarrow Q$ finite or $Q$ infinite f.g. torsion
ii) $H \leq f G \Rightarrow H$ f.g.i.t. by 6.11 \[ (H \text{ f.g. } \Rightarrow G \text{ f.g.)} \]
for $H \leq f G$

iii) $G$ is not virtually solvable.

Proof

iii) If $H$ is solvable then $H' \leq H$, f.g., torsion, abelian, infinite.

Thus $H'$ is f.g.i.t. by ii) and solvable. So continue until $H^n = 1$.

2. Burnside Problem

If $G$ is f.g. and $F \leq K$ such that $\forall g \in G$, $g^K = e$
(bounded torsion) then can $G$ be infinite?

Let $FB(n, k) = \langle x_1, \ldots, x_n \mid w^k = e \forall w \in F \rangle$

Then a group $G$ is n-gen and $g^K = e \forall g \in G$

$\Rightarrow FB(n, k) \rightarrow G$. so (2) says:

Do there exist $n, k$ for which $FB(n, k)$ is infinite?
3. Restricted Burnside Problem

Can $G$ in (2) be infinite and residually finite?

$\Rightarrow$ $\mathbb{F}_p(n, k)$ infinite by 8.3.

1. Isoda (1964): yes, $\exists$ infinite $p$-groups (every element has order $p^k$ for some $k$).

Schrage-Auchta (2011):
Let $p$ be some fixed prime.

Definition 9.2

In $F_n$, the $p$-value $v_p(w)$ of $w \in F_n$ is

$max \{ k : w = u^{p^k}, u \in F_n \}$

Definition 9.3

The $p$-deficiency ($p$-def.) of a presentation $\langle x_1, \ldots, x_n \mid r_1, r_2, r_3, \ldots \rangle$ is $n - \sum_{i=1}^{\infty} \frac{1}{p^{v_p(r_i)}}$

if it converges.

Lemma 9.4

Suppose that $F$ acts on $\emptyset$ and $S \triangleleft F$ with $[F : S] = p$.

For $x \in X$, if $\exists g \in \text{Stab}_F(x) \setminus S$ then $\text{Ord}_F(x) = \text{Ord}_S(x)$

Proof

Because $S$ in $F$ is maximal.

We have $S \triangleleft \text{Stab}_F(x) = F$, so for $f(x) \in \text{Ord}_F(x)$,

set $F = S$ and then $f(x) = s \cdot f(x) \in \text{Ord}_S(x)$.

\[ \square \]

Theorem 9.5

For any prime $p$, $n \geq 2$, $\exists$ infinite $n$-generated $p$-group

which is residually finite.
Proof
Suppose that \( \langle x_1, \ldots, x_n | r_1, r_2, \ldots \rangle \) is a presentation of \( G \) with \( p\text{-def}(G) \geq 1 \) we define \( G = F / R \)

\( \Phi : G \to C_p : \) We must have \( V_p(r_i) = 0 \) for at most \( n-1 \) relations, otherwise \( p\text{-def}(G) \leq n-1 \)

By (4.8), if \( \text{span} \{ \overline{r_i} \} \neq (1_{F_p})^n \) then \( G \to C_p \), but if \( V_p(r_i) \geq 1 \) then \( \overline{r_i} = 0 \in (1_{F_p})^n \) so \( \text{dim span} \{ \overline{r_i} \} < n \) so \( \Theta : G \to C_p \)

Thus, let \( N = \ker \Theta \) and set \( N = S / R \) for \( R \leq S \leq F_n \)
with \( [F_n : S] = p \). By the proof of (6.11), \( N \) is generated by \( p(n-1)+1 \) elements and \( R = \langle \langle t_i^r, t_i^{-r} \mid i \in N, 0 \leq i \leq p-1 \rangle \rangle \)
give relations, where \( \{e, t_1, \ldots, t_p\} \) is a transversal for \( S \) in \( F_n \) if \( t \not\in S \).

What is the \( p\text{-def} \)? Take one relation \( r = r_i \) in \( P \), and set \( k = V_p(r) \), so \( r = w^{p^k} \), for \( w \in F_n \). We have two cases:

a) If \( w \not\in S \):
By (9.4) with action conjugacy, \( x = r_i \) and \( f = w \) (commute), we get
\( \text{Conj. Class}_{F_n}(r) = \text{Conj. Class}_S (r) \).
So \( \langle \langle r_i, t_i^r, t_i^{-r} \rangle \rangle = \langle \langle r_i \rangle \rangle_S \)
Now \( r = (w^{p^k})^{p^k} \) for \( w^{p^k} \in S \)

b) If \( w \in S \):
\( t_i^r t_i^{-r} = (t_i^w t_i^{-r})^{p^k} \)
This tells us that \( R = \langle \langle r_i \rangle \rangle_{F_n} \) if \( V_p(r_i) \) in \( S \) or \( \in F_n \) \( r_i \in S \) if \( V_p(r_i) \)

So this presentation \( G \) for \( N \) has \( p\text{-def} \).
\[
p(n-1)+1 - \sum_{n=1}^{\infty} \frac{p}{n \cdot v_p(n)} = p \left( p^{\text{def}}(p) - 1 \right) + 1
\]

ii) If \( p^{\text{def}}(p) \geq 1 \), then \( G \) is infinite:

Since we get \( p^{\text{def}}(q) > 1 \), so \( N \to C_p \), so repeat to get \( G > N_1 > N_2 > \ldots \)

Now list the non-identity elements of \( F_n \) as \( \{ w_1, w_2, w_3, \ldots \} \)

and set \( P = \langle x_1, \ldots, x_n \mid w_1^p, w_2^p, w_3^p, \ldots \rangle \)

then \( p^{\text{def}}(P) \geq 1 \). So \( G \) is infinite and a \( p \)-group and finitely generated. What about residual finiteness?

By 8.3, \( G_{R_n} \) residually finite, a \( p \)-group, f.g.

Is it infinite? \( N_1, N_2, \ldots \to R_n \), so \( G_{R_n} \) is infinite.

\textbf{END OF COURSE MATERIAL}

2. \( FB(n,2) \) abelian, \( FB(n, \frac{3}{6}) \) finite (1940-50)

\( FB(n, 5) \) open. Novikov, Adyan (1970s) \( FB(n, k) \) infinite for all odd \( k \geq 668 \)

Ol'Shanski (82): A large prime \( p \), f.g. \( G \) such that if \( I < H < G \), then \( H \cong C_p \), so \( G \) has max!

3. No! By Zelmanov (Fields, 1994) and others.

Question:

\( \exists \, f.p. \) infinite torsion groups
Exam

Friday 30th May. Choose 3 questions from 5. Question out of 33

Definition 8.1

$G$ is residually finite if $\bigcap_{H \in \mathcal{P} G} H = 1$

Proposition 8.2

The following are equivalent:

i) $G$ is residually finite.

ii) $\bigcap_{N \in \mathcal{P} G} N = 1$

iii) $\forall g \in G \setminus \{e\}$, $\exists$ a homomorphism $\Theta$ onto $F$, a finite group such that $\Theta(g) \neq e$.

iv) $\forall g_1, \ldots, g_n \in G \setminus \{e\}, \Theta \rightarrow F$, finite, with $\Theta(g_i) \neq e$.

Proof

i) $\Rightarrow$ ii) Clear.

ii) $\Rightarrow$ iii) By 6.8. "useful theorem"

iii) $\Leftrightarrow$ ii) as for $g \in G \setminus \{e\}$, take $N = \ker \Theta$ or $\Theta : G \rightarrow G$

where $g \notin N$ such an $N$ exists by ii), with $N \in \mathcal{P} G$

ii) $\Rightarrow$ iv) via $\Theta : G \rightarrow \bigcap_{N \in \mathcal{P} G} N$ and Poincaré,

whereas iv) $\Rightarrow$ iii) is clear.

So finite groups and $\mathbb{Z}$ are residually finite, but $\mathbb{Q}$ and the Higman group are not.

An infinite residually finite group has infinitely many finite index subgroups (Poincaré) of arbitrarily high index by 6.11: if both $H = \bigcap_{i=1}^n H_n$, $J \triangleleft G$
$[H : HnJ \subseteq [G : H]$ implies containment?

$[G : H] = [G : HnJ] \iff H = J$ to take.

$H_{n+1} = H$ with $h \in H \setminus J$, continue, and index $\to \infty$.

*Aside*

Topological groups: Given a group $G$, we expect the discrete topology. A more interesting topology defines basic open sets to be cosets for $N$, $N \triangleleft G$. This is the profinite topology. Hausdorff $\iff G$ residually finite.

Indiscrete $\iff$ No proper finite index subgroups.

Discrete $\iff G$ finite. 

**Lemma 8.3**

If $R_a = \bigcap_{N \triangleleft G} N$, then $G/R_a$ is residually finite.

**Proof**

Normal finite index subgroups of $G/R_a$ are $N/R_a$ for $R_a \leq N \triangleleft G$, but for $g \not\in R_a$, we have $R_a \leq N \triangleleft G$ with $g \not\in N$. 

Note that if $G \twoheadrightarrow Q$ is residually finite, then $Q$ factors through $G/R_a$ as $Q \cap R_a \leq R_a$.

**Proposition 8.4**

i) $G$ residually finite, $H \leq G \Rightarrow H$ residually finite

ii) $H$ residually finite, $H \triangleleft G \Rightarrow G$ residually finite

iii) $G, H$ residually finite $\Rightarrow G \times H$ residually finite and

if $G$ is finitely generated, then $G \times H$ is residually finite.
Proof

i) $R_\alpha = \bigcap_{i \in L} L_i$ from 8.2 and $R_\alpha \cdot H \supseteq R_H$ by 6.(iii)

ii) For $H \leq G$ we have $L \leq H \rightarrow L \leq G$, so $R_\alpha \leq R_H$.

But $R_\alpha \supsetneq R_H$ by i).

iii) For $(g, h) \neq id$, take $\Theta_1 : G \rightarrow F_1$, $\Theta_2 : H \rightarrow F_2$,

$\Theta_1(g)$, $\Theta_2(h)$ not both $e$. Then $\Theta_1 \times \Theta_2 : G \times H \rightarrow F_1 \times F_2$,

$(g, h) \rightarrow (\Theta_1(g), \Theta_2(h))$, $(g, h) \neq (e, e)$.

For $G \times H$, we have $\Theta : G \times H \rightarrow H$ with $\Theta(gh) = h$, so

$\exists \psi$ with $\psi \Theta(gh) = \psi(gh) \neq e$ in some finite $F$. unless $h = e$.

Now, take $L \leq G$ with $g \notin L$, $(g \in G \setminus \ker \Theta$, some $g$), and

$C \leq L$, characteristic in $G$ by 6.10. Then $C \leq G \times H$.

Thus $C \leq G \times H$. As $G \times H = I$, we have $g \notin CH \leq G \times H$.

Corollary 8.5

$G$ virtually polycyclic $\Rightarrow G$ residually finite.

Proof

We have $H \leq G$ with $H$ polycyclic by 6.16.

Now suppose that if $M/N = Z$, $N$ finitely generated and residually finite, then $M \cong N \times Z$ by 3.5, so

$M$ residually finite by 8.4.iii).

Thus $H$ is residually finite and $G$ is too by 8.4.ii)

Theorem 8.6

Free groups are residually finite.
Proof 1
Recall 3.23, that $F = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $G = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \in SL_2(\mathbb{Z})$ freely generate $F_2$. Given a reduced word ($\neq \emptyset$) $w \in F_2$, we have $w(F, G) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Take a large prime $p$, $p > \max |a-1|, |d-1|, |b|, |c|$. Then,
$\Theta : SL_2(\mathbb{Z}) \to SL_2(\mathbb{F}_p)$ (reduction mod $p$).

$SL_2(\mathbb{F}_p)$ is finite and $\Theta(w(F, G)) \neq e$. So $F_2$ is residually finite. By alternative characterization of residual finiteness.

Now $F_0 < F_2$ so these are residually finite. For $w (\neq \emptyset) \in F(x)$, where $x = \{ x_i : i \in I \}$ only $x_i, \ldots, x_{i'}$ appear in $w$.

So we have $\Theta : F(x) \to F_k = F(x_{i_1}, \ldots, x_{i_n})$
given by sending the rest of $x$ to $e$ and extending.

Now $\Theta(w) \neq e$, so we now have $\Psi : F_k \to \text{finite group}$ with $\Psi \Theta(w) \neq e$.

Proof 2
$F_2$ free on $a, b$. We will create a reduced word $w$ using $a, b, A = a^{-1}$, $B = b^{-1}$ (formal inverses)

$w = A b A R S B A A b A$

Let $f : F_2 \to SL(2)$ (or $\mu_1$ if length $n$) be given by

<table>
<thead>
<tr>
<th>$ $</th>
<th>$1$</th>
<th>$2$</th>
<th>$3$</th>
<th>$4$</th>
<th>$5$</th>
<th>$6$</th>
<th>$7$</th>
<th>$8$</th>
<th>$9$</th>
<th>$10$</th>
<th>$11$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a$</td>
<td>$1$</td>
<td>$2$</td>
<td>$3$</td>
<td>$4$</td>
<td>$5$</td>
<td>$6$</td>
<td>$7$</td>
<td>$8$</td>
<td>$9$</td>
<td>$10$</td>
<td>$11$</td>
</tr>
<tr>
<td>$b$</td>
<td>$6$</td>
<td>$1$</td>
<td>$3$</td>
<td>$4$</td>
<td>$5$</td>
<td>$6$</td>
<td>$7$</td>
<td>$10$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Partial Functions. Each $i \to i \pm 1$.

Is this injective?
Suppose e.g. \( f(b)(9) = 10 = f(b)(11) \)?

Then \( B^{10} b^9 \) reduced.

\( f(a), f(b) \in S(11) \).

Now by the universal property, we can extend to a homomorphism \( f : F_2 \rightarrow S(11) \), \( f(a)(11) = 11 \), so \( f(w) \neq \text{id} \).

\[
\begin{array}{cccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\
9 & 11 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
6 & 11 & 3 & 1 & 2 & 4 & 5 & 6 & 7 & 10 & 8 & 9
\end{array}
\]
Theorem 8.6

a) $G_1, G_2$ residually finite $\Rightarrow G_1 \times G_2$ residually finite

Proof

First, suppose $G_1, G_2$ are finite. Given a reduced sequence $g_1, g_2, \ldots, g_n \in G_1 \times G_2$ of length $n > 1$, let

$$X_n = \{ g \in G_1 \times G_2 \mid 0 \leq \text{length}(g) \leq n \}$$

a finite set. Because $G_1, G_2$ finite, define an action on $X_n$ via the following:

- If $r \in G_2$ then $r(g_1, \ldots, g_n) = \left\{ \begin{array}{ll} r g_1 \ldots g_n & \text{if} \text{ length}(r) \leq n \\ g_1 \ldots g_n & \text{otherwise} \end{array} \right.$

We have a homomorphism $G_2 \rightarrow S(X_n)$

- Similarly, we have a homomorphism $G_1 \rightarrow S(X_n)$

We can extend this to a homomorphism $G_1 \times G_2 \rightarrow S(X_n)$, a finite group. (finite because $X_n$ is a finite set)

Now, done by 8.2, definition of res. fin. with homomorphisms.

Now $g_1, g_2, \ldots, g_n(\emptyset) = g_1, g_2, \ldots, g_n \neq \emptyset$ so that $g_1, g_2, \ldots, g_n$ is non-trivial in $S(X_n)$ (acts non-trivially on $\emptyset$)

For $g_1, g_2, \ldots, g_n$ in general $G_1 \times G_2$, choose $N_i \triangleleft G_i, N_2 \triangleleft G_2$ such that $g_1, \ldots, g_n \in N_1 \cup N_2$.

This extends to $G_1 \times G_2 \rightarrow \left( \frac{G_i}{N_i} \right) \ast \left( \frac{G_2}{N_2} \right)$ and the image of $g_1, \ldots, g_n$ is reduced to has length $n$.

Hopfian Groups

Definition 8.7

A group $G$ is Hopfian if every surjective endomorphism
Theorem 8.8 (Malcev 1940)
A finitely generated, residually finite group $G$ is Hopfian.

Proof
For $\Theta : G \to G$ and $H \leq G$ with index $n$, $\Theta^{-1}(H)$ has index $n$ too, and if $\Theta^{-1}(H_1) = \Theta^{-1}(H_2)$, then
$$\Theta^{-1}(H_1) = H_1 = H_2.$$ So the pullback map is injective. 

Corollary 8.9
If $g_1, \ldots, g_n$ generate the free group $F_n$, then they freely generate $F_n$.

Proof
If $F_n$ is residually finite, then $Z \leq F_n$ is residually finite.

If $F_n$ is Hopfian, so if $w(g_1, \ldots, g_n) = e$ in $F(x_1, \ldots, x_n)$, then $\Theta : F(g_1, \ldots, g_n) \to F(x_1, \ldots, x_n)$ by:

(given a symbol $g_i$), $g_i \mapsto (\text{its image on RHS})$.

This is injective since it hits the generating set.
Hence this is injective \( (a, b) \), so \( w = 0 \)

An infinite simple group is Hopfian but not residually finite.

**Baumslag–Solitar Group**

**Definition 8.10**

The Baumslag–Solitar group \( B_{m,n} = \langle a, b \mid b a^n b^{-1} = a^m \rangle \)

for \( m, n \neq 0 \). So these include \( B_{1,1} = \mathbb{Z} \times \mathbb{Z} \),

\( B_{1,\infty} = \text{fundamental group of the Klein bottle} \).

We can also change \( m, n \) and keep the group the same:

\( B_{m,n} \cong B_{r,m} \), \( B_{m,n} \cong B_{n,m} \).

They are HNN extensions \( \langle a \rangle *_{\phi} \) with \( \phi : \langle a^m \rangle \to \langle a^n \rangle \).

**Proposition 8.11**

\( B_{m,n} \) is solvable if \( |m| = 1 \) or \( |n| = 1 \) and contains \( F_2 \) otherwise.

**Proof**

If \( |m| = 1 \) or \( |n| = 1 \), then WLOG we have \( B_{1,n} \), and this is solvable, just as in 5.11 \((B_{1,2})\). Otherwise,

\( x \) a domain or image of \( \phi \). So for any reduced word

\( w(x, y) \in F_2 \), we have \( w(t, aba^{-1}) \) is a reduced sequence

in an HNN extension. So this is \( \neq e \) by Bruckner's Lemma.

**Theorem 8.12**

\( B_{2,3} \) is not Hopfian.

**Proof**

Let \( \theta(t) = t \), \( \theta(a) = a^2 \). This is a homomorphism as it
preserves the relation: \( \Theta(ta^2t^{-1}) = (ta^2t^{-1})^2, \Theta(a^3) = a^6 \)

Is this injective? Yes, because \( tat^{-1}a^{-1} \rightarrow ta^2t^{-1}a^{-2} = a \)

What about the kernel? \( \Theta([tat^{-1}, a]) = [a^3, a^2] = e \)

\([tat^{-1}, a] = tat^{-1}a^{-1}t^{-1}a^{-1} \) is a reduced sequence, so \( t \neq e \) by Britton's Lemma.

**Theorem 8.13**

If f.g. solvable \( G \) which is not f.p.

**Proof**

Consider \( G = B_{2,13} \) and \( \Theta \) as above. Set \( k_i = \ker \Theta_i \triangleleft G \) with \( k_i < k_{i+1} \). For \( y \neq e \), \( \Theta(y) = e \), we have

\[ y = \Theta_i(x) \Rightarrow \Theta_i \text{ is injective, so } x \in k_i + 1 \backslash k_i \]

So \( G(k_i) = Q \) is not f.p. by 4.4, since \( x_i \neq 0 \), then

\[ Q = \langle a, b \mid S \rangle \text{ with } S = \langle S_i, \ldots, S_k \rangle \text{ in } G, \]

so all in \( K_n \). But \( x \in k_{n+1} \backslash k_n \) is \( e \) in \( Q \), but not in \( S \subseteq K_n \).

For some \( N \) large enough

Now \( G' \) is generated by \( tat^{-i} \) for \( i \in \mathbb{Z} \). But

\[ \Theta_i(tat^{-i}) = tat^{-i} = a^{3i} \] \( i \geq 0 \) which commutes with \( \Theta_i(a) \). This gives \( [tat^{-i}, a] \rightarrow e \) in \( Q \) as \( it \) is in \( \ker \Theta_i \), as well as \( [Eiat^{-i}, tat^{-i}] \) by conjugacy (set \( i = i + k \)). Thus \( \Theta(G') = Q' \) is abelian so

\[ Q'' = I \]

became \( \Theta \) (generator for \( G' \))

and thee commute.
9. The Generalized Burnside Problem

Examples of Torsion Groups

Finite groups $F$, infinite $FxFx\ldots$, example 1.5.

None are f.g.

1. Generalized Burnside Problem (1902)

Do there exist infinite f.g. torsion groups?

Lemma 9.1

If $G$ is such a group then

i) $G \rightarrow \mathbb{Q}$ $\Rightarrow$ $\mathbb{Q}$ finite or $\mathbb{Q}$ infinite f.g. torsion

ii) $H \leq G \Rightarrow H$ f.g.i.t. by 6.11 $\Leftrightarrow \left( H \text{ f.g. } \Leftrightarrow G \text{ f.g. } \right)$ for $H \leq G$

iii) $G$ is not virtually soluble.

Proof

iii) If $H$ is soluble then $H'\text{ f.g., torsion, abelian, infinite.}$

Thus $H'$ is f.g.i.t. by ii) and soluble. So continue until $H^{(n)} = 1$.

2. Burnside Problem

If $G$ is f.g. and $F$ such that $\forall g \in G, g^k = e$ (bounded torsion) then can $G$ be infinite?

Let $FB(n,k) = \langle x_1, \ldots, x_n | w^k = e \forall w \in F_n \rangle$

Then a group $G$ is n-gen and $g^k = e \forall g \in G$

$\Leftrightarrow FB(n,k) \rightarrow G$, so (2) says:

Do there exist $n,k$ for which $FB(n,k)$ is infinite?
3. Restricted Burnside Problem

Can \( G \) in (2) be infinite and residually finite

\[ \iff \FB(n, k) \text{ infinite by \& 3.} \]

1. Golod (1964): yes, \( \exists \) infinite f.g. p-groups (every element has order \( p^k \) for some \( k \)).

Schlage - Puchta (2011):

Let \( p \) be some fixed prime.

Definition 9.2

In \( F_n \), the \( p \)-value \( \nu_p(w) \) of \( w \in F_n \) is

\[ \max \{ k : w = w^{p^k}, \ u \in F_n \} \]

Definition 9.3

The \( p \)-deficiency (\( p \)-def) of a presentation

\[ \langle x_1, \ldots, x_n \mid r_1, r_2, r_3, \ldots \rangle \] is

\[ n - \sum_{i=1}^{\infty} \frac{1}{p^{\nu_p(r_i)}} \]

if it converges.

Lemma 9.4

Suppose that \( F \) acts on \( \Theta \) and \( S \leq F \) with \( [F : S] = p \).

For \( x \in \Theta \) if \( f \in \Stab_F(x) \setminus S \) then \( \Orb_F(x) = \Orb_S(x) \).

Proof:

Because \( S \) in \( F \) is maximal.

We have \( \Stab_F(x) = F \), so for \( f(x) \in \Orb_F(x) \),

set \( f = \sigma \) then \( f(x) = \sigma(x) \in \Orb_S(x) \).

\[ \square \]

Theorem 9.5

For any prime \( p \), \( n \geq 2 \), \( \exists \) infinite \( n \)-generated p-group

which is residually finite.
Proof.
Suppose that \( \langle x_1, \ldots, x_n \mid r_1, r_2, \ldots \rangle \) is a presentation \( P \) with \( p\text{-def}(P) \leq 1 \) defines \( G = F_n \).

1. \( \phi: G \to C_p \) : We must have \( V_p(r_i) = 0 \) for at most \( n-1 \) relators, otherwise \( p\text{-def}(P) = n-1 \). By (4.8), if 
\[ \text{span} \{ \tilde{r}_i \} \neq (F_n)^n \text{ then } G \to C_p, \text{ but if } V_p(r_i) > 1 \text{ then } \tilde{r}_i = 0 \in (F_n)^n \text{ so dim span} \{ \tilde{r}_i \} = n, \text{ so } \theta: G \to C_p \]

Thus, let \( N = \ker \theta \) and set \( N = \frac{S}{R} \) for \( R = S \cup F_n \) with \( [F_n : S] = p \).

By the proof of (6.11), \( N \) is generated by \( p(n-1) + 1 \) elements and \( R = \langle \langle t^i, t^{-i} \mid i \in N, 0 \leq i \leq p-1 \rangle \rangle \) give relators, where \( t^i, t^{-i} \) is a transversal for \( S \) in \( F_n \) if \( t \notin S \).

What is the \( p\text{-def} \)? Take one relator \( r = t^j \) in \( P \), and set \( k = V_p(r) \), so \( r = w^k \) for \( w \in F_n \). We have two cases:

a) If \( w \notin S \):

By (9.4) with action conjugacy, \( x = r, A = w \) (conmute), we get 
\[ \text{Conj. Class}_{F_n}(r) = \text{Conj. Class}_{S}(r). \]

So \( \langle \langle t^i, t^{-i} \rangle \rangle_s = \langle \langle t^i \rangle \rangle_s \)

Now \( r = (w^k)^{t^i \cdots t^{-i}} \) for \( w^k \in S \)

b) If \( w \in S \):

\[ t^i t^{-j} = \frac{(t^i w t^{-j})^k}{w^k} \in S \cup F_n \]

This tells us that \( R = \langle \langle r_i \rangle \mid V_p(r_i) = \frac{V_p(r_i) - 1}{w^k} \in S \cup F_n \rangle \)

So this presentation \( Q \) for \( N \) has \( p\text{-def} \).
\[ p^{(n-1)} + 1 - \sum_{n=1}^{\infty} \frac{p^n}{N_0(n)} = p \left( \rho_{\text{def}}(p) - 1 \right) + 1 \]

ii) If \( \rho_{\text{def}}(p) > 1 \), then \( G \) is infinite.

Since we get \( p_{\text{def}}(Q) > 1 \), so \( N \to C_p \), so repeat to get \( G > N_1 > N_2 > \ldots \)

Now list the non-identity elements of \( F_n \) as \( \{w_1, w_2, w_3, \ldots \} \)

and set \( P = \langle x_1, \ldots, x_n \mid w_1^p, w_2^{p^2}, w_3^{p^3}, \ldots \rangle \)

then \( \rho_{\text{def}}(p) > 1 \). So \( G \) is infinite and a \( p \)-group and finitely generated. What about residual finiteness?

By \& 3, \( G \) is residually finite, a \( p \)-group, \& \( G \)

Is it infinite? \( N_1, N_2, \ldots \geq R_p \), so \( G \) is infinite. \( \square \)

END OF COURSE MATERIAL

2. FB(n, 2) abelian, FB(n, \( \frac{3}{5} \)) finite (1940-50)

FB(n, 5) open. Novikov, Adyan (1970s) FB(n, k) infinite for all odd \( k \), \( k \geq 6 \).

Ol'shanski (82): \( p \) large primes \( \rho \), \( p \rho \) such that if \( I < H < G \), then \( H \approx C_p \), so \( G \) has max.

W. No! By Zelmanov (Fields, 1994) and others.

Question

\( F? \) \( p \)-infinite torsion groups